AD665140

Air Force Office of Scientific Research
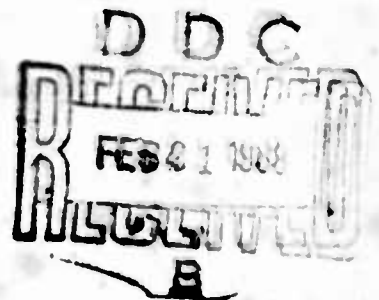
Grant AF-AFOSR-127-67

AN ELASTO-PLASTIC ANALYSIS OF CIRCULAR RINGS

WITH TEMPERATURE DEPENDENT YIELD STRESS

by

N. T. Patel and L. W. Hu

Department of Engineering Mechanics
THE PENNSYLVANIA STATE UNIVERSITY
University Park, Pennsylvania

November 1967

ABSTRACT

The elastic-plastic behavior of a ring under steady state radial temperature gradient is analyzed. The material is assumed to be elastic-perfectly plastic and its yield stress in simple shear to be a continuous and general monotonically decreasing positive function of temperature. Modified Tresca's yield condition and the associated flow rules are used.

NOTATIONS

a = inner boundary of an elastic region in the ring

b = outer boundary of an elastic region in the ring

c = inner boundary of the ring

d = outer boundary of the ring

E = Young's modulus

h = half the thickness of the ring

k = yield stress in simple shear

$k(T)$ = k as a function of temperature

$k_r, k_\rho, k_\eta, k_c, k_d$ = value of k at the place denoted by the subscript

k_0 = k at $0°F$

p_a, p_b = pressures at the elastic boundaries a and b respectively

r, θ, z = cylindrical coordinates

r_m = place in an elastic region where $\dfrac{d\sigma_r}{dr} = 0$

r_c = place in an elastic region where $\sigma_\theta = 0$

T = temperature in $°F$

T_r, T_c, T_d, T_a, T_b = T at the place denoted by the subscript

u, w = radial and axial displacements respectively

α = coefficient of thermal expansion

β = absolute value of slope of a linear function $k(T)$

$\Delta T = T_c - T_d$

$\Delta T^e = T_a - T_b$

ΔT^i = ΔT for incipience of yielding

ΔT^c = ΔT for complete yielding of the ring

$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ = radial, tangential and axial strains respectively

$\epsilon_r', \epsilon_\theta', \epsilon_z'$ = elastic parts of the radial, tangential and axial strains respectively

$\epsilon_r'', \epsilon_\theta'', \epsilon_z''$ = plastic parts of the radial, tangential and axial strains respectively

η, ρ = elastic-plastic interfaces $(\eta > \rho)$

ν = Poisson's ratio

$\sigma_r, \sigma_\theta, \sigma_z$ = radial, tangential and axial stresses respectively

$\sigma_{r\rho}, \sigma_{r\eta}$ = σ_r at $r = \rho$ or η respectively

$\sigma_{\theta\rho}, \sigma_{\theta\eta}$ = σ_θ at $r = \rho$ or η respectively

An Elasto-plastic Analysis of Circular Rings With

Temperature Dependent Yield Stress

by

N. T. Patel and L. W. Hu

1. Introduction.

During the last two decades a number of thermoelastic-plastic
studies on symmetric problems have been made. For instance, the
solutions have been obtained for the steady state temperature problems
of thick-walled spheres by Cowper[1]*, Johnson and Mellor[2] and
Rogozinski[3], thick-walled cylinders by Bland[4] and Kammash, Murch
and Naghdi[5] and thin rings by Wilhoit[6]. Problems have been
studied for transient temperatures in plates by Yuksel[7], Landau,
Weiner and Zwicky[8], and Mendelson and Spero[9], solid disks by
Parkus[10], cylinders by Landau and Zwicky[11], and half spaces by Lee
and Jaunzemis[12]. In most of these investigations, the mechanical and
thermal parameters--modulus of elasticity E, Poisson's ratio ν, yield
stress in simple shear k, coefficient of expansion α, thermal conductivity
and specific heat--have been assumed to be independent of temperature and
stresses. Only in a few papers[3, 8, 9, 11, 12] has the thermal dependence
of the yield stress in shear k(T) been taken into account. In reference
[3] thermal dependence of thermal conductivity has been also included.
However, non-isothermal yield conditions more general than those in-
cluding thermal dependence through k(T) have not been used. A theory

---

*Numbers in square brackets refer to the list of references at the end
of the paper.

on the flow rules associated with general nonisothermal yield condition has been suggested by Prager[13] in 1958. To this theory Boley and Weiner[14] have given a plausible explanation and attempts have been made to apply the theory in particular using k(T) in Tresca's or von Mises' yield condition.

The purpose of the present study is to investigate in detail the influence of thermal dependence of yield stress on elastic-plastic stresses and deformation of a thin, finite, annular ring subjected to a steady state radial temperature gradient with traction free boundaries. For the case of isothermal Tresca's yield condition, a stress solution to this problem has been given by Wilhoit[6]. In this paper, the Tresca's yield condition, but modified in that the yield stress in simple shear k is a continuous, general, monotonically decreasing positive function of T, i.e. $k = k(T) > 0$, $\frac{dk}{dT} < 0$, is used. Flow rules associated with this yield condition[13,14] are used to investigate the deformation. For illustration, a solution for linear k(T) is presented. The material is assumed to be elastic-nonstrainhardening and other physical parameters are assumed to be independent of temperature and stresses.

In the present problem the loading is thermal and is considered only due to a difference in the temperatures at the two boundaries of the ring. The ring is considered to be initially stress and strain-free and at a uniform temperature. A complete monotonically increasing loading program is assumed and unloading is not to be considered here. Two cases depending upon whether the inner temperature is higher or lower than the outer temperature become different and require

individual treatment. In each case the loading may be considered to proceed in two ways. In one instance, the loading is assumed to proceed such that the temperature at a boundary is changed extremely slowly and in infinitesimal increments while the temperature at the other boundary is held constant. The temperature distribution may be assumed to be in steady state all the time. On the other hand, if it is felt that this kind of loading history is really not continuous because the transient states between the steady states are neglected, one may instead imagine the present problem applicable to problems in which there are heat sources everywhere on each face keeping at any time and leading during loading the ring through steady state temperature conditions.

Three points may be noted: (1) The temperature at a boundary may be changed by either heating or cooling. (2) During a loading program, the instantaneous temperatures at either boundaries may alternately be held constant while the other is subjected to variation. For instance, first the temperature at the inside boundary may be raised for some time while the temperature at the outside boundary is held constant and then the temperature at the outside boundary may be lowered holding the temperature at the inside boundary constant. (3) However, the case of varying the temperatures at both boundaries simultaneously is not included in this analysis.

The fact that stresses in the plastic regions are statically determinate except in one special case simplifies the determination of the stresses in the elastic regions. The displacement solution is obtained and provides a justification to the _a priori_ assumption of

infinitesimal strains at high temperatures or at high difference in the
boundary temperatures.

2. Fundamental Equations.

Consider a circular ring of inside radius c and outside radius d
with thickness 2h, as shown in Figure 1. Let the coordinate system be
cylindrical (r,θ,z) with the origin on the central axis of the ring and
at the midpoint of its thickness. The following sixteen equations are
valid in an elastic region.

The radial stress σ_r and the tangential stress σ_θ are related by
the equilibrium equation:

$$r \frac{d\sigma_r}{dr} = \sigma_\theta - \sigma_r \tag{1}$$

The strain-displacement equations are:

$$\epsilon_r = \frac{du}{dr} \tag{2}$$

$$\epsilon_\theta = \frac{u}{r} \tag{3}$$

and

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{4}$$

where ϵ_r, ϵ_θ and ϵ_z are the radial, tangential and axial strains
respectively, and $u(r)$ and $w(r,z)$ are the radial and axial displace-
ments respectively.

Let the temperatures at the inner boundary $r = c$ and the outer
boundary $r = d$ be T_c and T_d respectively. It is well known that, in
the case of radial steady state temperature gradient, the temperature
distribution must satisfy the Laplace's equation

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{5}$$

The solution of this equation is

$$T = T_c - \frac{\Delta T}{\ell n \frac{d}{c}} \ell n \frac{r}{c} \tag{6}$$

or

$$T = T_d + \frac{\Delta T}{\ell n \frac{d}{c}} \ell n \frac{d}{r} \tag{7}$$

where

$$\Delta T = T_c - T_d \quad . \tag{8}$$

ΔT may be positive or negative.

Assuming that the present problem is a plane stress one, axial stress

$$\sigma_z = 0 \tag{9}$$

and

$$\varepsilon'_r = \frac{\sigma_r}{E} - \frac{\nu \sigma_\theta}{E} + \alpha T \tag{10}$$

$$\varepsilon'_\theta = \frac{\sigma_\theta}{E} - \frac{\nu \sigma_r}{E} + \alpha T \tag{11}$$

and

$$\varepsilon'_z = - \frac{\nu \sigma_r}{E} - \frac{\nu \sigma_\theta}{E} + \alpha T \tag{12}$$

where E and α are the modulus of elasticity and coefficient of thermal expansion respectively. Single and double primes will be used to denote the elastic and plastic parts of the strain respectively.

Consider an arbitrary elastic region $a \leq r \leq b$ where $a \geq c$ and $b \leq d$ with temperatures T_a and T_b at the boundaries $r = a$ and $r = b$ respectively. Elimination of u from Eqs. 2 and 3 gives a relation

between ϵ_r' and ϵ_θ'. This relation, after substituting Eqs. 10, 11, 1 and 6 into it, reduces to a simple second order differential equation of σ_r for $a \leq r \leq b$

$$\frac{d^2\sigma_r}{dr^2} + \frac{3}{r}\frac{d\sigma_r}{dr} - \frac{2}{r^2}\frac{E\alpha\Delta T^e}{\ln \frac{b^2}{a^2}} = 0 \tag{13}$$

where

$$\Delta T^e = T_a - T_b = \frac{\Delta T}{\ln \frac{d}{c}} \ln \frac{b}{a} \tag{14}$$

Corresponding to ΔT, ΔT^e may be positive or negative. If the pressures on the elastic boundaries are p_a at $r = a$ and p_b at $r = b$, the solution of Eq. 13 and Eq. 1 give:

$$\sigma_r = - p_a \frac{\frac{b^2}{r^2} - 1}{\frac{b^2}{a^2} - 1} - p_b \frac{\frac{b^2}{a^2} - \frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} + \frac{E\Delta\alpha T^e}{2}\left[\frac{\frac{b^2}{r^2} - 1}{\frac{b^2}{a^2} - 1} - \frac{\ln \frac{b^2}{r^2}}{\ln \frac{b^2}{a^2}}\right] \tag{15}$$

$$\sigma_\theta = p_a \frac{\frac{b^2}{r^2} + 1}{\frac{b^2}{a^2} - 1} - p_b \frac{\frac{b^2}{a^2} + \frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} - \frac{E\Delta\alpha T^e}{2}\left[\frac{\frac{b^2}{r^2} + 1}{\frac{b^2}{a^2} - 1} + \frac{\ln \frac{b^2}{r^2} - 2}{\ln \frac{b^2}{a^2}}\right] \tag{16}$$

Equations 1 to 12 are valid in a plastic region if the shape of the ring does not change significantly during plastic deformation. By Tresca's yield condition, in a plastic region $\tau \equiv \frac{1}{2}$ max. $[|\sigma_r|, |\sigma_\theta|, |\sigma_\theta - \sigma_r|] = k$, where τ is the maximum shear stress and k is the yield

stress in shear. As mentioned in the introduction, k is assumed to be a continuous, general, monotonically decreasing, positive and finite valued function of temperature.

$$0 < k = k(T) < \infty \quad \text{and} \quad \frac{dk}{dT} < 0 \tag{17}$$

Whereas $k(T)$ must be continuous everywhere, $\frac{dk(T)}{dT}$ may be continuous piecewise.

In view of Eqs. 6 and 7, when ΔT and either T_c or T_d are constants, k can be assumed directly such that

$$0 < k = k(r) < \infty, \quad \frac{dk}{dr} > 0 \quad \text{if} \quad \Delta T > 0 \quad \text{and} \quad \frac{dk}{dr} < 0 \quad \text{if} \quad \Delta T < 0 \tag{18}$$

When r and ΔT are both to be considered independent variables, the yield stress k can be expressed as

$$k = k(r,\Delta T), \text{ either } T_c \text{ or } T_d \text{ is held constant}, \tag{19}$$

and

$$\frac{\partial k}{\partial \Delta T} \geq 0 \quad \text{when } T_c \text{ held constant}$$

or

$$\frac{\partial k}{\partial \Delta T} \leq 0 \quad \text{when } T_d \text{ held constant} \tag{20}$$

$\frac{\partial k}{\partial \Delta T}$ vanishes at $r = c$ when T_c is held constant and at $r = d$ when T_d is held constant. Which of the functions for k given by Eqs. 17,18 and 19 are to be used will be clear in the analysis depending upon the considerations being made.

For an illustration $k(T)$ is taken to be

$$k = k_0 - \beta T \tag{21}$$

where k_0 is yield stress in shear at $0°F$ and β is an appropriate positive constant. By Eq. 6 or 7, this equation reduces to

$$k = k_c + \frac{\beta \Delta T}{\ln \frac{d}{c}} \ln \frac{r}{c} \tag{22}$$

or

$$k = k_d - \frac{\beta \Delta T}{\ln \frac{d}{c}} \ln \frac{d}{r} \, , \tag{23}$$

where k_c and k_d are the yield stresses in shear at the boundaries $r = c$ and $r = d$ respectively.

3. <u>Behavior of Stresses in an Elastic Region ($\Delta T \gtreqless 0$).</u>

Consider an arbitrary elastic region $a \leq r \leq b$ ($a \geq c$, $b \leq d$) of the ring. The expressions for stress components in Eqs. 15 and 16 are continuous and differentiable.

$$\frac{d\sigma_r}{dr} = \frac{1}{r} \left[\frac{2 \frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} (p_a - p_b) + \left(\frac{-\frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} + \frac{1}{\ln \frac{b^2}{a^2}} \right) E\alpha \Delta T^e \right] \tag{24}$$

$$\frac{d\sigma_\theta}{dr} = \frac{1}{r} \left[\frac{2 \frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} (-p_a + p_b) + \left(\frac{\frac{b^2}{r^2}}{\frac{b^2}{a^2} - 1} + \frac{1}{\ln \frac{b^2}{a^2}} \right) E\alpha \Delta T^e \right] \tag{25}$$

$$\frac{d(\sigma_\theta - \sigma_r)}{dr} = \frac{2}{r}\left[\frac{2\frac{b^2}{r^2}}{\frac{b^2}{a^2}-1}(-p_a + p_b) + \left(\frac{\frac{b^2}{r^2}}{\frac{b^2}{a^2}-1}\right)E\alpha\Delta T^e\right] \tag{26}$$

By Eq. 24, $\frac{d\sigma_r}{dr}$ vanishes and hence σ_r assumes maximum or minimum at

$$r_m = b\left[\frac{2\ln\frac{b^2}{a^2}}{\frac{b^2}{a^2}-1}\left(\frac{-p_a + p_b}{E\alpha\Delta T^e} + \frac{1}{2}\right)\right]^{\frac{1}{2}} \tag{27}$$

r_m cannot exist, if $\frac{-p_a + p_b}{E\alpha\Delta T^e} \leq -\frac{1}{2}$; and if it does exist it is unique.

Now, if the inequalities

$$a < b\left[\frac{2\ln\frac{b^2}{a^2}}{\frac{b^2}{a^2}-1}\left(\frac{-p_a + p_b}{E\alpha\Delta T^e} + \frac{1}{2}\right)\right]^{\frac{1}{2}} < b \tag{28}$$

are satisfied by the conditions imposed on the boundaries of the region $a \leq r \leq b$, then the following three important conclusions can be drawn as to the nature of the stresses in the elastic region: (1) σ_r assumes the maximum value (if $\Delta T^e < 0$) or the minimum value (if $\Delta T^e > 0$) at only one place r_m in the region, (2) $\sigma_r = \sigma_\theta$ at r_m and (3) σ_θ and $(\sigma_\theta - \sigma_r)$ increase (if $\Delta T^e > 0$) or decrease (if $\Delta T^e > 0$) monotonically in the region. The first two conclusions result because the Inequalities 28 require r_m by Eq. 27 to be $a < r_m < b$, and at r_m, where

$\frac{d\sigma_r}{dr} = 0$, $\frac{d^2\sigma_r}{dr^2} \lessgtr 0$ depending upon $\Delta T^e \gtrless 0$ respectively by Eq. 13 and $\sigma_\theta = \sigma_r$ by Eq. 1. The third conclusion is due to the Eqs. 25 and 26 and the Inequality $\frac{-p_a + p_b}{E\alpha\Delta T^e} + \frac{1}{2} > 0$ implied by the first of the Inequalities 28. Figure 2 (a=c, b=d), Figure 5 (ρ=a, d=b) and Figure 6 (ρ=a, η=b) illustrate the above behavior of elastic stresses. The Inequalities 28, when rearranged, reduce to

$$\left[\frac{\frac{b^2}{a^2} - 1}{\frac{b^2}{a^2} \ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^e}{2} < -p_a + p_b < \left[\frac{\frac{b^2}{a^2} - 1}{\ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^{\cdot\cdot}}{2}, \text{ if } \Delta T^e > 0$$

(29)

or,

$$\left[\frac{\frac{b^2}{a^2} - 1}{\frac{b^2}{a^2} \ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^e}{2} > -p_a + p_b > \left[\frac{\frac{b^2}{a^2} - 1}{\ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^e}{2}, \text{ if } \Delta T^e < 0$$

(30)

In the following, three different sets of the boundary conditions are shown satisfying the Inequalities 29 (hence, Inequalities 28) when $\Delta T^e > 0$ (i.e. by Eq. 14, $\Delta T > 0$):

(1) $-p_a = -p_b = 0$. The Inequalities 29 to be satisfied reduce to

$$\left[\frac{\frac{b^2}{a^2} - 1}{\frac{b^2}{a^2} \ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^e}{2} < 0 < \left[\frac{\frac{b^2}{a^2} - 1}{\ln \frac{b^2}{a^2}} - 1\right] \frac{E\alpha\Delta T^e}{2}$$

(31)

It can be verified that $-1 < \dfrac{\dfrac{b^2}{a^2} - 1}{\dfrac{b^2}{a^2} \ell n \dfrac{b^2}{a^2}} - 1 < 0$ and $0 < \dfrac{\dfrac{b^2}{a^2} - 1}{\ell n \dfrac{b^2}{a^2}} - 1 < \infty$ for

$1 < \dfrac{b}{a} < \infty$. Figure 2 where $c = a$ and $d = b$ illustrates this case.

(2) $-p_a + p_b < 0$ and $(\sigma_\theta - \sigma_r) < 0$ at $r = a$, $-p_a + p_b < 0$ and the second of Inequalities 31 prove the second of Inequalities 29. Using the second condition that $(\sigma_\theta - \sigma_r) < 0$ at $r = a$ with Eqs. 15 and 16, the first of Inequalities 29 can be proved easily. Figure 5 where $\rho = a$ and $d = b$ illustrates this case.

(3) $\sigma_\theta - \sigma_r < 0$ at $r = a$ and $\sigma_\theta - \sigma_r > 0$ at $r - b$. Both these conditions with Eqs. 15 and 16 satisfy easily the Inequalities 29. Figure 6 where $\rho = a$ and $\eta = b$ illustrates this case.

For $\Delta T^e < 0$ (i.e. $\Delta T < 0$, by Eq. 14) each of the following four different sets of boundary conditions satisfy Inequalities 30 (hence Inequalities 28) just as above:

(1) $-p_a = -p_b = 0$ Figure 2.

(2) $-p_a + p_b > 0$ and $\sigma_\theta - \sigma_r > 0$ at $r = a$.

(3) $\sigma_\theta - \sigma_r > 0$ at $r = a$ and $\sigma_\theta - \sigma_r < 0$ at $r = b$.

(4) $-p_a + p_b < 0$ and $\sigma_\theta - \sigma_r < 0$ at $r = b$.

Thus, if a set of boundary conditions for an elastic region is one of the sets mentioned in this section, the elastic stresses would behave according to the conclusions made above and the incipience of yielding, new yielding, further yielding or progress of yielding would be easy to predict from that behavior as the loading progresses.

4. <u>Initial Yielding $(\Delta T \lessgtr 0)$.</u>

In an entirely elastic ring $c \leq r \leq d$ $(1 < \frac{c}{d} < \infty)$ with temperature gradient $(\Delta T = \Delta T^e \lessgtr 0)$, the stresses σ_r and σ_θ have distributions of the kind shown in Figure 2 as discussed in the previous section where $a = c$ and $b = d$. The distribution of the maximum shear stress $\tau = \frac{1}{2}$ maximum $[|\sigma_r|, |\sigma_\theta|, |\sigma_\theta - \sigma_r|]$ then follows to be a continuous function in three different pieces as shown in Figure 3 for $\Delta T > 0$ and Figure 4 for $\Delta T < 0$. $\tau = \frac{1}{2} |\sigma_\theta|$ in $a \leq r \leq r_m$ (at $r = r_m$, $\sigma_\theta = \sigma_r$ and $\frac{d\sigma_r}{dr} = 0$), $\tau = \frac{1}{2} |\sigma_r|$ in $r_m \leq r \leq r_c$ (at $r = r_c$, $\sigma_\theta = 0$), and $\tau = \frac{1}{2} |\sigma_\theta - \sigma_r|$ in $r_c \leq r \leq d$. τ decreases monotonically in $c \leq r \leq r_c$ and increases monotonically in $r_c \leq r \leq d$. At $r = c$, $\tau = \frac{1}{2} |\sigma_\theta|$; and also at $r = d$, $\tau = \frac{1}{2} |\sigma_r - \sigma_\theta| = \frac{1}{2} |\sigma_\theta|$. It can be shown from Eq. 16 that $|\sigma_\theta|$ at $r = c$, $|\sigma_{\theta c}|$ is always $(1 < \frac{d}{c} < \infty)$ greater than $|\sigma_{\theta d}|$, $|\sigma_\theta|$ at $r = d$. k is monotonic but increases if $\Delta T > 0$ and decreases if $\Delta T < 0$.

From Figure 3, it is clear that the incipience of yielding can occur only inside at $r = c$ for a certain $\Delta T(>0)$ as ΔT is being increased from zero while keeping one of the boundary temperatures constant. For $\Delta T > 0$, ΔT^i the value of ΔT for the initial yielding is given by the following equation obtained from the condition that $\frac{1}{2} |\sigma_{\theta c}| = k_c (\equiv k$ at $r = c)$.

$$\left(\frac{E\alpha}{2k_c} \right) \Delta T^i = \left[\frac{1}{\frac{d^2}{c^2} - 1} - \frac{1}{\ln \frac{d^2}{c^2}} + 1 \right]^{-1} \tag{32}$$

ΔT^1 is explicitly determinable in Eq. 31 if T_c and hence k_c are held

constant. ΔT^1 becomes implicitly determinable in Eq. 31 if T_d is held

constant; because k_c is then a function of T_d and ΔT. For simple,

linear $k(T)$ (Eq. 21) however

$$\frac{\beta \Delta T^1}{k_d} = \left[1 + \frac{\alpha E}{2\beta} \left(\frac{1}{\frac{d^2}{c^2} - 1} - \frac{1}{\ln \frac{d^2}{c^2}} + 1 \right) \right]^{-1} \tag{32a}$$

In the above discussion T_c or T_d need not necessarily be assumed held

constant all the time from the beginning; it is which of T_c or T_d that

is held constant during a step that initiates the yielding is important.

From Figure 4, it is clear that the incipience of yielding can

occur anywhere at one place or many places on account or the natures of

k and r. Hence for $\Delta T < 0$, the incipience of yielding is highly

dependent upon the relation $k(r)$ which in turn is dependent upon $k(T)$

and the loading history. No general equation such as Eq. 32 for ΔT^1

can be written. Thus general step by step analysis is prevented from

here for $\Delta T < 0$.

Complete step by step elastic-plastic analysis is carried out only

for the case of $\Delta T > 0$ in the following sections. A section at the end

discusses the analysis in general for the case of $\Delta T < 0$.

5. Inner Plastic Region $c \leq r \leq \rho$ ($\Delta T > 0$).

Incipience of yielding occurs at $r = c$ when $\sigma_\theta = -2k$. As the

loading continues a finite plastic region will develop inside. Subject

to condition $\sigma_\theta \leq \sigma_r \leq 0$ the yield condition

$$\sigma_\theta = -2k \tag{33}$$

Eq. 1, and the boundary condition: $\sigma_r = 0$ at $r = c$, determine

$$\sigma_r = -\frac{2}{r} \int_c^r k \, dr \tag{34}$$

Since $k > 0$ (Inequalities 17), $\sigma_\theta < 0$ and $\sigma_r \le 0$. Since k increases monotonically, the area under curve k from c to r, $\int_c^r k \, dr$ is less than the area of the rectangle enclosing the curve, $k_r(r - c)$, where k_r is the value of k at an arbitrary point r. Therefore $k_r r > k_r(r - c) > \int_c^r k \, dr$ and $k_r > \frac{1}{r} \int_c^r k \, dr$. It follows that

$$\sigma_\theta < \sigma_r \le 0 \tag{35}$$

Thus at any stage of loading the yield condition Eq. 33 will always be valid. It may be checked that any other form of the yield condition will not give an acceptable stress solution.

The condition of continuity of σ_r across the interface ρ and the condition: $\sigma_r = 0$ at $r = d$, allow to find the stresses in the elastic region $\rho \le r \le d$ easily from the general Eqs. 15 and 16 used with Eqs. 14 and 34.

$$\sigma_r = \left(-\frac{2}{\rho} \int_c^\rho k \, dr \right) \frac{\dfrac{d^2}{r^2} - 1}{\dfrac{d^2}{\rho^2} - 1} + \frac{E\alpha\Delta T}{2 \ln \dfrac{d}{c}} \ln \frac{d}{\rho} \left[\frac{\dfrac{d^2}{r^2} - 1}{\dfrac{d^2}{\rho^2} - 1} - \frac{\ln \dfrac{d^2}{r^2}}{\ln \dfrac{d^2}{\rho^2}} \right] \tag{36}$$

$$\sigma_\theta = \left(\frac{2}{\rho} \int_c^\rho k\ dr\right) \frac{\dfrac{d^2}{r^2} + 1}{\dfrac{d^2}{\rho^2} - 1} - \frac{E\alpha\Delta T}{2\ \ell n\ \dfrac{d}{c}}\ \ell n\ \frac{d}{\rho}\ \left(\frac{\dfrac{d^2}{r^2} + 1}{\dfrac{d^2}{\rho^2} - 1} + \frac{\ell n\ \dfrac{d^2}{r^2} - 2}{\ell n\ \dfrac{d^2}{\rho^2}}\right) \qquad (37)$$

Equations 36 and 37 will give complete stress distribution upon the determination of ρ. The use of continuity of σ_θ across ρ gives the following relation to determine ρ.

$$f(\Delta T,\rho) \equiv 2k_\rho + \left(\frac{2}{\rho} \int_c^\rho k\ dr\right) \frac{\dfrac{d^2}{\rho^2} + 1}{\dfrac{d^2}{\rho^2} - 1} + \frac{E\alpha\Delta T}{2\ \ell n\ \dfrac{d}{c}} \left[\left(-\ \ell n\ \frac{d^2}{\rho^2}\right)\left(\frac{\dfrac{d^2}{\rho^2}}{\dfrac{d^2}{\rho^2} - 1}\right) + 1\right] = 0.$$

$$(38)$$

This equation and Eq. 32 show that $\rho = c$ when $\Delta T = \Delta T^i$. As ΔT increases from ΔT^i, if $\dfrac{d\rho}{d\Delta T}$ is continuous -- piecewise if necessary, and

$$\frac{d\rho}{d\Delta T} > 0 \quad , \qquad (39)$$

then ρ increases with ΔT. Inequality 39 can be seen always satisfied in the following equation due to Inequality 20 if ΔT is increasing and T_d is constant. If T_c is held constant, Inequality 39 is satisfied definitely only when $\rho = c$ but may or may not be satisfied when $\rho > c$. Therefore, since Eq. 39 is satisfied in any case when $\rho = c$, after an incipience of yielding a small plastic region will definitely develop inside for a small increment in ΔT.

$$\frac{d\rho}{d\Delta T} = -\frac{\frac{\partial f}{\partial \Delta T}}{\frac{\partial f}{\partial \rho}} =$$

$$\frac{-2\frac{\partial}{\partial \Delta T}\left[k_\rho + \left(\frac{1}{\rho}\int_c^\rho k\,dr\right)\frac{\frac{d^2}{\rho^2}+1}{\frac{d^2}{\rho^2}-1}\right] + \frac{E\alpha}{2\ln\frac{d}{c}}\left[\left(\ln\frac{d^2}{\rho^2}\right)\frac{\frac{d^2}{\rho^2}}{\frac{d^2}{\rho^2}-1} - 1\right]}{\frac{2}{\rho}\left\{\rho\frac{\partial k_\rho}{\partial \rho} + \left(k_\rho - \frac{1}{\rho}\int_c^\rho k\,dr\right)\frac{\frac{d^2}{\rho^2}+1}{\frac{d^2}{\rho^2}-1} + \frac{\frac{4}{\rho}}{\frac{d^2}{\rho^2}-1}\int_c^\rho k\,dr + \frac{E\alpha\Delta T}{2\ln\frac{d}{c}}\left(1 - \frac{\ln\frac{d^2}{\rho^2}}{\frac{d^2}{\rho^2}-1}\right)\frac{\frac{d^2}{\rho^2}}{\frac{d^2}{\rho^2}-1}\right\}}$$

(40)

The denominator and the second term of the numerator are positive. The first term of the numerator will be positive if T_d is constant; but if T_c is constant, it will be zero if $\rho = c$ and negative if $\rho > c$, by Inequality 20. If ρ decreases at certain ΔT before new yielding occurs, it means that the plastic region is undergoing unloading and becoming elastic. At such point the above analysis of stresses must be modified and the analysis cannot be continued along the following lines.

For the elastic region $\rho \le r \le d$, the stresses at the boundaries ρ and d (Inequalities 35) satisfy the second set of conditions (for $\Delta T > 0$) discussed in the section on the behavior of stresses in an elastic region. σ_r, σ_θ, and hence τ, should therefore be distributed as shown in Figure 5. It is evident that the yield condition is not violated in the elastic region and further yielding continues from ρ until a second plastic region starts to develop outside if the

difference between k_d and τ_d (τ at $r = d$ ($= \frac{1}{2} |\sigma_\theta - \sigma_r|$ at $r = d$) $=$ $\frac{1}{2} \sigma_{\theta d}$) decreases to zero, i.e. $d(k_d - \tau_d) < 0$ or $d\tau_d > dk_d \geq 0$ as $k_d - \tau_d$ is approaching zero. For $d\Delta T > 0$ and $d\rho > 0$, $dk_d \geq 0$ because k_d remains constant if T_d = constant and increases if T_c = constant; but $d\tau_d$ ($= \frac{1}{2} d\sigma_{\theta d}$), when examined from Eq. 37, cannot be definitely shown to be positive, negative or zero. Therefore, as the loading progresses it is possible that $\tau_d = k_d$ may not be satisfied at all and the second yielding may never occur. $\tau_d = k_d$ by Eq. 37 is

$$2k_d = \left[\frac{2}{\rho}\int_c^\rho k\, dr\right] \frac{2}{\frac{d^2}{\rho^2} - 1} + \frac{E\alpha\Delta T}{2\, \ell n\, \frac{d}{c}} \left\{1 - \frac{\ell n\, \frac{d^2}{\rho^2}}{\frac{d^2}{\rho^2} - 1}\right\} \tag{41}$$

Equations 38 and 41 determine ρ and ΔT when the second yielding starts outside. The first two terms of Eq. 38 are positive; the third becomes definitely nonnegative if $\frac{d}{\rho} \leq 1$; hence if the solutions $(\rho, \Delta T(<\infty))$ of Eqs. 38 and 41 exist, ρ must be such that $\rho < d$.

If the second yielding never occurs and ρ keeps increasing with ΔT, then finite ΔT cannot satisfy Eq. 38 as $\frac{d}{\rho} \to 1$. Therefore, in such a case the ring can become completely plastic only if ΔT approaches infinity without violating the assumptions regarding k.

Before ΔT reaches its limit which may have been established due to the limitations assumed for k, such as $k(r)$ remains always positive, it is possible that the second yielding outside may never be caused to occur. For example, for $\frac{d}{c} = 10$, linear $k(T)$ (Eq. 23, k_d held constant) and $\frac{E\alpha}{4\beta} = 3$ the second yielding is predicted by Eqs. 38 and 41 when $\frac{\beta\Delta T}{k_d}$

has exceeded its limit 1 and become 1.169 requiring $k(r)$ to be negative in $1 \leq \frac{r}{c} \leq 1.38$.

6. <u>Outer Plastic Region</u> $\eta \leq r \leq d$ $(\Delta T > 0)$.

A finite plastic region may form outside after the second yielding starts outside due to $k_d = \frac{1}{2} |\sigma_\theta - \sigma_r|$ at $r = d$. The expressions for stresses in the inner plastic region remain the same (Eqs. 33 and 34); the stresses in the outer plastic region and in the central elastic region are now to be determined.

In the outer plastic region $\eta \leq r \leq d$, the Eq. 1, subject to condition that $\sigma_r \leq 0 \leq \sigma_\theta$ the yield condition

$$\sigma_\theta - \sigma_r = 2k \tag{42}$$

and the boundary condition: $\sigma_r = 0$ at $r = d$, determine

$$\sigma_r = -2 \int_r^d \frac{k}{r} \, dr \tag{43}$$

and

$$\sigma_\theta = 2k - 2 \int_r^d \frac{k}{r} \, dr \tag{44}$$

Since $k > 0$ (Inequalities 17), for $r \leq d$,

$$\sigma_r \leq 0 \tag{45}$$

$\sigma_\theta = 2k_d$ at $r = d$ and $\frac{d\sigma_\theta}{dr} = 2\left(\frac{dk}{dr} + \frac{k}{r}\right) > 0$; hence, monotonically increasing σ_θ will be positive in $\eta \leq r \leq d$, if the inequality $\sigma_{\theta\eta}$ (σ_θ at $r = \eta$) > 0, i.e.

$$k_\eta - \int_\eta^d \frac{k}{r} \, dr > 0 \qquad (46)$$

is always satisfied. This condition will be shown always satisfied at the end of this section. Thus the yield condition Eq. 42 remains always valid. It may be checked that any other form of the yield condition will not give an acceptable stress solution.

The condition of continuity of σ_r across the interfaces ρ and η allow to find the stresses in the elastic region $\rho \le r \le \eta$ easily from the general Eqs. 15 and 16 used with Eqs. 14, 34 and 43.

$$\sigma_r = \left(-\frac{2}{\rho} \int_c^\rho k \, dr \right) \frac{\dfrac{\eta^2}{r^2} - 1}{\dfrac{\eta^2}{\rho^2} - 1} - \left(2 \int_\eta^d \frac{k}{r} \, dr \right) \frac{\dfrac{\eta^2}{\rho^2} - \dfrac{\eta^2}{r^2}}{\dfrac{\eta^2}{\rho^2} - 1}$$

$$+ \frac{E\alpha\Delta T}{2 \ln \dfrac{d}{c}} \ln \frac{\eta}{\rho} \left[\frac{\dfrac{\eta^2}{r^2} - 1}{\dfrac{\eta^2}{\rho^2} - 1} - \frac{\ln \dfrac{r^2}{r^2}}{\ln \dfrac{\eta^2}{\rho^2}} \right] \qquad (47)$$

$$\sigma_\theta = \left(\frac{2}{\rho} \int_c^\rho k \, dr \right) \frac{\dfrac{\eta^2}{r^2} + 1}{\dfrac{\eta^2}{\rho^2} - 1} - \left(2 \int_\eta^d \frac{k}{r} \, dr \right) \frac{\dfrac{\eta^2}{\rho^2} + \dfrac{\eta^2}{r^2}}{\dfrac{\eta^2}{\rho^2} - 1}$$

$$- \frac{E\alpha\Delta T}{2 \ln \dfrac{d}{c}} \ln \frac{\eta}{\rho} \left[\frac{\dfrac{\eta^2}{r^2} + 1}{\dfrac{\eta^2}{\rho^2} - 1} + \frac{\ln \dfrac{\eta^2}{r^2} - 2}{\ln \dfrac{\eta^2}{\rho^2}} \right] \qquad (48)$$

Equations 47 and 48 will give complete stress distribution upon the determination of ρ and η. ρ and η can be determined by using the following two relations which are conditions of continuity of σ_θ across ρ and η.

$$-2k_\rho = \left(\frac{2}{\rho}\int_c^\rho kdr\right)\frac{\frac{\eta^2}{\rho^2}+1}{\frac{\eta^2}{\rho^2}-1} - \left(2\int_\eta^d \frac{k}{r}\,dr\right)\frac{2\frac{\eta^2}{\rho^2}}{\frac{\eta^2}{\rho^2}-1} - \frac{E\alpha\Delta T}{2\ln\frac{d}{c}}\ln\frac{\eta^2}{\rho^2}\left[\frac{\frac{\eta^2}{\rho^2}}{\frac{\eta^2}{\rho^2}-1} - \frac{1}{\ln\frac{\eta^2}{\rho^2}}\right]$$

(49)

$$2k_\eta - 2\int_\eta^d \frac{k}{r}\,dr =$$

$$\left(\frac{2}{\rho}\int_c^\rho kdr\right)\frac{2}{\frac{\eta^2}{\rho^2}-1} - \left(2\int_\eta^d \frac{k}{r}\,dr\right)\frac{\frac{\eta^2}{\rho^2}+1}{\frac{\eta^2}{\rho^2}-1} - \frac{E\alpha\Delta T}{2\ln\frac{d}{c}}\ln\frac{\eta^2}{\rho^2}\left[\frac{1}{\frac{\eta^2}{\rho^2}-1} - \frac{1}{\ln\frac{\eta^2}{\rho^2}}\right]$$

(50)

The Eqs. 49 and 50 may be reduced to two simpler equations by multiplying Eq. 50 by $\frac{\eta^2}{\rho^2}$ and then subtracting Eq. 49 from it and obtaining

$$f(\rho,\eta,\Delta T) \equiv \left[k_\rho - \frac{1}{\rho}\int_c^\rho k\,dr + \frac{E\alpha\Delta T}{4\ln\frac{d}{c}}\right]\rho^2 - \left[\frac{E\alpha\Delta T}{4\ln\frac{d}{c}} - k_\eta\right]\eta^2 = 0 \quad (51)$$

and by subtracting Eq. 49 from Eq. 50 and obtaining

$$g(\rho,\eta,\Delta T) \equiv k_\rho + \frac{1}{\rho}\int_c^\rho k\,dr + \frac{E\alpha\Delta T}{4\ell n\frac{d}{c}}\ell n\rho^2 - \frac{E\alpha\Delta T}{4\ell n\frac{d}{c}}\ell n\eta^2 + k_\eta - 2\int_\eta^d \frac{k}{r}\,dr = 0 \tag{52}$$

It is interesting to note that Eqs. 51 and 52 are nothing but the following two equations:

$$-\sigma_{\theta\rho} + \sigma_{r\rho} + (\sigma_{\theta\eta} - \sigma_{r\eta})\frac{\eta^2}{\rho^2} = \frac{E\alpha\Delta T}{2\ell n\frac{d}{c}}\left(\frac{\eta^2}{\rho^2} - 1\right) \tag{53}$$

$$-\sigma_{\theta\rho} - \sigma_{r\rho} + \sigma_{\theta\eta} + \sigma_{r\eta} = \frac{E\alpha\Delta T}{2\ell n\frac{d}{c}}\ell n\frac{\eta^2}{\rho^2} \tag{54}$$

The stresses σ_r and σ_θ with subscript ρ or η are to be evaluated at $r = \rho$ or $r = \eta$ respectively.

For the elastic region $\rho \leq r \leq \eta$, the stresses at the boundaries $r = \rho$ and $r = \eta$ (Inequalities 35, 45 and 46) satisfy the third set of conditions (for $\Delta T > 0$) discussed in the section on the behavior of stresses in an elastic region. σ_r, σ_θ and, hence, τ should therefore be distributed as shown in Figure 6. It is evident that the yield condition will not be violated in the elastic region, third plastic region will not develop and the yielding will continue further at the interfaces.

Equations 51 and 52 determine ρ and η , but they are general and implicit, and cannot be solved in closed form; they require numerical solution which could be handled easily by the use of a digital computer. Therefore, it is important to establish the existence of the solution and the procedure to obtain it.

The Eqs. 51 and 52 can be reduced to Eqs. 38 and 41 when $\eta = d$. It was shown in the previous section that the plastic region may develop outside only if the second yielding starts at $r = d$ for $\rho(< d)$ and ΔT which satisfy the Eqs. 38 and 41. Therefore, Eqs. 51 and 52 have a unique solution $\rho, \eta(= d)$ for ΔT which starts yielding outside, and will have unique solution as ΔT increases if $\dfrac{d\rho}{d\Delta T}$ and $\dfrac{d\eta}{d\Delta T}$ are continuous -- piecewise if necessary, and

$$\frac{d\rho}{d\Delta T} \geq 0 \tag{55}$$

$$\frac{d\eta}{d\Delta T} \leq 0 \tag{56}$$

These conditions are also required for the validity of the adopted analysis which assumes that the plastic regions do not unload as the loading progresses. Figure 7 provides an illustration that it is possible to have cases which do not always satisfy Inequalities 55 and 56. It is evident in Figure 7, which is drawn for linear $k(T)$ (Eq. 23, k_d held constant) from Eqs. 51 and 52, that (for $\dfrac{d}{c} = 5$ and $\dfrac{E\alpha}{4\beta} = 100$) η starts to increase at A. Analysis must be modified from the point A to include the unloading of outer plastic region.

It is assumed that the Inequalities 55 and 56 are always satisfied. The solution of Eqs. 51 and 52 may be obtained from the intersection of the curves f and g defined by the functions f and g respectively in the $\eta\rho$ plane, as shown in Figure 8. The slopes of the curves f and g are given respectively by

$$\frac{\partial \rho}{\partial \eta}^f = \frac{-\dfrac{\partial f}{\partial \eta}}{\dfrac{\partial f}{\partial \rho}} = \frac{\eta}{\rho}\,\frac{\dfrac{E\alpha\Delta T}{4\,\ell n\,\dfrac{d}{c}} - k_\eta - \dfrac{\eta}{2}\dfrac{\partial k_\eta}{\partial \eta}}{\dfrac{1}{2}\left(k_\rho - \dfrac{1}{\rho}\displaystyle\int_c^\rho k\,dr\right) + \dfrac{E\alpha\Delta T}{4\,\ell n\,\dfrac{d}{c}} + \dfrac{\rho}{2}\dfrac{\partial k_\rho}{\partial \rho}} \tag{57}$$

$$\frac{\partial \rho}{\partial \eta}^g = \frac{-\dfrac{\partial g}{\partial \eta}}{\dfrac{\partial g}{\partial \rho}} = \frac{\rho}{\eta}\,\frac{\dfrac{E\alpha\Delta T}{4\ell n\,\dfrac{d}{c}} - k_\eta - \dfrac{\eta}{2}\dfrac{\partial k_\eta}{\partial \eta}}{\dfrac{1}{2}\left(k_\rho - \dfrac{1}{\rho}\displaystyle\int_c^\rho k\,dr\right) + \dfrac{E\alpha\Delta T}{4\ell n\,\dfrac{d}{c}} + \dfrac{\rho}{2}\dfrac{\partial k_\rho}{\partial \rho}} \tag{58}$$

and at the intersection point the following relations are satisfied.

$$\frac{\partial f}{\partial \eta} = \eta^2 \frac{\partial g}{\partial \eta} \tag{59a}$$

$$\frac{\partial f}{\partial \rho} = \rho^2 \frac{\partial g}{\partial \rho} \tag{59b}$$

$$\frac{\partial \rho}{\partial \eta}^f = \frac{\eta^2}{\rho^2}\frac{\partial \rho}{\partial \eta}^g \tag{59c}$$

The Eqs. 59 result really on account of the nature of Eqs. 51 and 52 as expressed in Eqs. 53 and 54, and Eq. 1. The curves f and g may intersect each other at many points, if between two consecutive inter-section points $\left(\dfrac{\partial \rho}{\partial \eta}^f - \dfrac{\partial \rho}{\partial \eta}^g\right)$ becomes zero. This is not possible unless, in view of Eq. 59c and the positiveness of denominators of Eqs. 57 or 58,

$$\frac{E\alpha\Delta T}{4\,\ell n\,\dfrac{d}{c}} - k_\eta - \frac{\eta}{2}\frac{\partial k_\eta}{\partial \eta} = 0 \tag{60}$$

Therefore, if the left side of Eq. 60 is positive (negative) for $\rho \leq \eta \leq d$, the curves f and g increase (decrease) monotonically and intersect at only one point giving a unique solution. If Eq. 60 is satisfied for $\rho \leq \eta \leq d$, multiple solutions may arise; an appropriate solution must then be chosen from the consideration that ρ and η should be continuous for the neighboring values of ΔT, as shown in Figure 8.

$\frac{d\rho}{d\Delta T}$ and $\frac{d\eta}{d\Delta T}$ may be obtained as follows by solving two equations: $\frac{df}{d\Delta T} = 0$ and $\frac{dg}{d\Delta T} = 0$, and using Eqs. 59.

$$\left(\frac{1}{\rho^2} - \frac{1}{\eta^2}\right)\left(\frac{\partial f}{\partial \rho}\right)\left(\frac{d\rho}{d\Delta T}\right) = \frac{1}{\eta^2}\frac{\partial f}{\partial \Delta T} - \frac{\partial g}{\partial \Delta T} \tag{61}$$

$$-\left(\frac{1}{\rho^2} - \frac{1}{\eta^2}\right)\left(\frac{d\eta}{d\Delta T}\right) = \frac{1}{-\frac{\partial f}{\partial \eta}}\left(\frac{\partial g}{\partial \Delta T} - \frac{1}{\rho^2}\frac{\partial f}{\partial \Delta T}\right) \tag{62}$$

In Eq. 61, $\frac{\partial f}{\partial \rho}$ (which is twice the denominator in Eq. 57) is always positive. In order that the Inequalities 55 and 56 be satisfied, the right sides of Eqs. 61 and 62 must be positive; which (unlike Eq. 40) are not always guaranteed by either of the cases in which T_c or T_d is held constant, or when $\eta = d$.

To show that in $\eta \leq r \leq d$ $\sigma_\theta > 0$, i.e., Inequality 46 always holds, Eqs. 51 and 52 may be used to obtain the following equation by eliminating $\frac{E\alpha\Delta T}{4 \ln \frac{d}{c}}$ between them.

$$\left(k_\eta - \int_\eta^d \frac{k}{r}\,dr\right) =$$

$$= \left\{k_\eta\left(\frac{n^2}{\rho^2} - \frac{\ln\frac{n^2}{\rho^2}}{\frac{n^2}{\rho^2} - 1}\right) + \left(k_\rho - \frac{1}{\rho}\int_c^\rho k\,dr\right)\left(1 - \frac{\ln\frac{n^2}{\rho^2}}{\frac{n^2}{\rho^2} - 1}\right)\right\}\left\{\frac{\frac{n^2}{\rho^2} - 1}{\ln\frac{n^2}{\rho^2}} + \frac{\ln\frac{n^2}{\rho^2}}{\frac{n^2}{\rho^2} - 1}\right\}^{-1}$$

$$(63)$$

Clearly, as required, the left side is positive because the right side is positive on account of the Inequalities 17, 35 and $1 < \frac{n}{\rho} < \infty$.

7. Total Yielding $c < r \leq d$ $(\Delta T > 0)$.

If the Inequalities 55 and 56 are satisfied, the interfaces ρ and η move toward each other as ΔT increases. The ring then tends to become totally plastic as $\frac{n}{\rho} \to 1$; and Eq. 53 (or Eq. 54) leads to the result that $\dfrac{E\alpha\Delta T^c}{2\ln\frac{d}{c}} = \lim_{\frac{n}{\rho}\to 1}\dfrac{\sigma_{\theta\eta} - \sigma_{\theta\rho}}{\frac{n^2}{\rho^2} - 1} = \infty$, where ΔT^c is ΔT required for complete yielding, because the continuity of σ_r requires $\lim_{\frac{n}{\rho}\to 1}(\sigma_{r\rho} - \sigma_{r\eta})$ $= 0$ and the Inequalities 35 and 46 require $0 < \sigma_{\theta\eta} - \sigma_{\theta\rho} < \infty$. It is assumed that the assumptions made for k are not violated. Hence, just as it is the case with other thermal problems, ΔT is required to be infinite in the present problem for complete yielding of the ring. However, the assumptions made for the function k (Inequalities 17 or 18) will impose limitation on ΔT in a problem which is being solved from actual physical data.

8. <u>Displacement and Strains ($\Delta T > 0$).</u>

It is assumed that the plastic regions develop and expand mono-
tonically with increasing loading (ΔT) as discussed in the previous
sections. Plastic deformation is assumed incompressible. The yield
condition $\tau = k(T)$ is nonisothermal; but, since it is very simple in
form, the flow rules which have been derived from the consideration
of a general nonisothermal yield condition[13,14] give the same
relations among the ratios of the plastic strain increments as those
customarily used for the isothermal yield condition[15], as shown in
Figure 9.

<u>Inner Plastic Region</u> ($c \leq r \leq \rho$). The flow rules require,
corresponding to line EF ($\sigma_\theta = -2k$) in Tresca hexagon (Figure 9),

$$\dot{\epsilon}_r'' : \dot{\epsilon}_\theta'' : \dot{\epsilon}_z'' :: 0 : -1 : 1 \tag{64}$$

Therefore, $\epsilon_r = \epsilon_r'$; or by Eqs. 2 and 10

$$E \frac{du}{dr} = \sigma_r - \nu\sigma_\theta + E\alpha T \tag{65}$$

Integration of Eq. 65 gives the displacement u with an arbitrary
constant of integration u_ρ --the displacement at $r = \rho$.

$$E(u - u_\rho) = -\int_r^\rho \sigma_r dr + \nu \int_r^\rho \sigma_\theta dr - E\alpha \int_r^\rho T\,dr \tag{66}$$

u_ρ can be determined by utilizing the fact that the strains at $r = \rho$
must be elastic.

$$Eu_\rho = E\rho\epsilon_{\theta\rho} = E\rho\epsilon'_{\theta\rho} = \rho(\sigma_{\theta\rho} - \nu\sigma_{r\rho} + E\alpha T_\rho) \tag{67}$$

σ_r and σ_θ in Eq. 66 and their values at $r = \rho$ in Eq. 67 are given by Eqs. 33 and 34; and T in Eq. 66 and its value at $r = \rho$ in Eq. 67 are given by Eq. 6 (T_c constant) or by Eq. 7 (T_d constant).

Outer Plastic Region ($\eta \leq r \leq d$). The flow rules require, corresponding to line AB ($\sigma_\theta - \sigma_r = 2k$) in Tresca hexagon (Figure 9),

$$\dot{\epsilon}'_r : \dot{\epsilon}''_\theta : \dot{\epsilon}''_z :: -1 : 1 : 0 \tag{68}$$

Therefore, $\epsilon_r + \epsilon_\theta = \epsilon'_r + \epsilon'_\theta$ or by Eqs. 2, 3, 10 and 11,

$$E\left(\frac{du}{dr} + \frac{u}{r}\right) = (1-\nu)(\sigma_\theta + \sigma_r) + 2E\alpha T \tag{69}$$

Integration of Eq. 69 gives the displacement u with an arbitrary constant of integration u_η -- the displacement at $r = \eta$.

$$E(ru - \eta u_\eta) = (1-\nu) \int_\eta^r (\sigma_\theta + \sigma_r)dr + 2E\alpha \int_\eta^r T \, dr \tag{70}$$

u_η can be determined by utilizing the fact that the strains at $r = \eta$ must be elastic.

$$Eu_\eta = E\eta\epsilon_{\theta\eta} = E\eta\epsilon'_{\theta\eta} = \eta(\sigma_{\theta\eta} - \nu\sigma_{r\eta} + E\alpha T_\eta) \tag{71}$$

σ_r and σ_θ in Eq. 70 and their values at $r = \eta$ in Eq. 71 are given by Eqs. 43 and 44; and T in Eq. 70 and its value at $r = \eta$ in Eq. 71 are given by Eq. 6 (T_c constant) or by Eq. 7 (T_d constant).

Central Elastic Region ($\rho \leq r \leq \eta$). $\epsilon_\theta = \epsilon_\theta'$ or by Eqs. 3 and 11 the displacement u is given by

$$Eu = r(\sigma_\theta - \nu\sigma_r + E\alpha T) \tag{72}$$

Equations 47 and 48 give the stresses σ_r and σ_θ and Eq. 6 or 7 gives T to be used in Eq. 72. When the region $\rho \leq r \leq d$ is elastic, Eq. 72 has stresses σ_r and σ_θ given by Eqs. 36 and 37.

It may be observed that the displacement field in the plastic regions is obtainable from the elastic strains of those regions (Eqs. 65 and 69), the interface displacements depend upon the pure elastic deformation of the central elastic region, and the elastic strains in the plastic regions cannot be excessive due to the assumption of perfect plasticity which requires the stresses to be bounded. Therefore, the strains in the ring will be of the order of magnitude of elastic strains and no danger of large deformations or thickening of the ring arises. Thus a priori assumption of infinitesimal strains in the analysis carried out is justified.

Figures 10 and 11 show the displacement and strains at $r = c$ and $r = d$ respectively versus ΔT in the case of linear $k(T)$ (Eq. 23). The conditions: $\dot{\epsilon}_\theta'' < 0$ in $c \leq r \leq \rho$ and $\dot{\epsilon}_\theta'' > 0$ in $\eta \leq r \leq d$ are required by Eqs. 64 and 68, but are not apparently seen satisfied by Eqs. 65 and 69 respectively from which displacements and strains are completely obtained. Figures 10 and 11 confirm respectively that $\dot{\epsilon}_\theta'' < 0$ at $r = c$ and $\dot{\epsilon}_\theta'' > 0$ at $r = d$. These inequalities have also been confirmed for various points in the plastic regions by the digital computer results.

9. <u>Solution when k Linear Function of T ($\Delta T > 0$, T_d = Constant)</u>.

It is assumed that T_d, and hence k_d, is held constant. Equation 23 in the form of

$$\frac{k}{k_d} = 1 - \frac{\beta \Delta T}{k_d} \; \frac{\ln \frac{d}{r}}{\ln \frac{d}{c}} \tag{73}$$

is used where

$$0 < \frac{\beta \Delta T}{k_d} < 1 \tag{74}$$

in order to satisfy the condition $k > 0$.

<u>Plastic Region Inside</u>. Stresses in inner plastic region $c \leq r \leq \rho$ are, by Eqs. 33 and 34 with Eq. 73,

$$\frac{\sigma_\theta}{2k_d} = 1 - \frac{\beta \Delta T}{k_d} \; \frac{\ln \frac{d}{r}}{\ln \frac{d}{c}} \tag{75}$$

$$\frac{\sigma_r}{2k_d} = -1 + \frac{c}{r} + \frac{\beta \Delta T}{k_d} \frac{1}{\ln \frac{d}{c}} \left[\left(1 - \frac{c}{r}\right)\left(1 + \ln \frac{d}{c}\right) - \ln \frac{r}{c} \right] \tag{76}$$

Stresses in the elastic region $\rho \leq r \leq d$ corresponding to Eqs. 36 and 37 may be obtained easily by using Eqs. 75 and 76. Relation between ρ and ΔT, corresponding to Eq. 38, is

$$\frac{\beta \Delta T}{k_d} = \frac{\left[\frac{c}{\rho}\left(1 + \frac{\rho^2}{d^2}\right) - 2\right] \ln \frac{d}{c}}{\frac{E\alpha}{4\beta}\left(1 - \frac{\rho^2}{d^2} - \ln \frac{d^2}{\rho^2}\right) + \ln \frac{\rho^2}{c^2} - \ln \frac{d}{c}\left(1 - \frac{\rho^2}{d^2}\right) - \left(1 + \frac{\rho^2}{d^2}\right)\left(1 - \frac{c}{\rho}\right)\left(1 + \ln \frac{d}{c}\right)} \tag{77}$$

Using this equation, it is possible to obtain the stresses in the elastic region $\rho \leq r \leq d$ in terms of ρ exclusively.

$$\frac{\sigma_r}{2k_d} = \frac{c}{2\rho}\left(1 + \frac{\rho^2}{r^2}\right) - 1 +$$

$$-\left[\frac{c}{2\rho}\left(1+\frac{\rho^2}{d^2}\right)-1\right]\frac{\frac{E\alpha}{4\beta}\left(1-\frac{\rho^2}{r^2}-\ln\frac{r^2}{\rho^2}\right)+\ln\frac{\rho^2}{c^2}-\ln\frac{d}{c}\left(1-\frac{\rho^2}{r^2}\right)-\left(1+\frac{\rho^2}{r^2}\right)\left(1-\frac{c}{\rho}\right)\left(1+\ln\frac{d}{c}\right)}{\frac{E\alpha}{4\beta}\left(1-\frac{\rho^2}{d^2}-\ln\frac{d^2}{\rho^2}\right)+\ln\frac{\rho^2}{c^2}-\ln\frac{d}{c}\left(1-\frac{\rho^2}{d^2}\right)-\left(1+\frac{\rho^2}{d^2}\right)\left(1-\frac{c}{\rho}\right)\left(1+\ln\frac{d}{c}\right)}$$

$$(78)$$

$$\frac{\sigma_\theta}{2k_d} = \frac{c}{2\rho}\left(1 - \frac{\rho^2}{r^2}\right) - 1 +$$

$$-\left[\frac{c}{2\rho}\left(1+\frac{\rho^2}{d^2}\right)-1\right]\frac{\frac{E\alpha}{4\beta}\left(1-\frac{\rho^2}{r^2}+\ln\frac{r^2}{\rho^2}\right)-\ln\frac{\rho^2}{c^2}+\ln\frac{d}{c}\left(1+\frac{\rho^2}{r^2}\right)+\left(1-\frac{\rho^2}{r^2}\right)\left(1-\frac{c}{\rho}\right)\left(1+\ln\frac{d}{c}\right)}{\frac{E\alpha}{4\beta}\left(1-\frac{\rho^2}{d^2}-\ln\frac{d^2}{\rho^2}\right)+\ln\frac{\rho^2}{c^2}-\ln\frac{d}{c}\left(1-\frac{\rho^2}{d^2}\right)-\left(1+\frac{\rho^2}{d^2}\right)\left(1-\frac{c}{\rho}\right)\left(1+\ln\frac{d}{c}\right)}$$

$$(79)$$

The value of ρ, for which the yielding starts outside at $r = d$, may be obtained from Eqs. 79 and 73 by $\sigma_{\theta d} = 2k_d$, i.e. $\frac{\sigma_{\theta d}}{2k_d} = 1$, or

$$\frac{E\alpha}{4\beta}\left[\left(\frac{c}{\rho}-3\right)\left(1-\frac{\rho^2}{d^2}\right)+\left(\frac{c}{\rho}\frac{\rho^2}{d^2}+1\right)\ln\frac{d^2}{\rho^2}\right]-\ln\frac{\rho^2}{c^2}\left(\frac{c}{\rho}\frac{\rho^2}{d^2}+1\right)$$

$$+\ln\frac{d}{c}\left(2\frac{c}{\rho}\frac{\rho^2}{d^2}+1-3\frac{\rho^2}{d^2}\right)+\left(1-\frac{c}{\rho}\right)\left(1+\ln\frac{d}{c}\right)\left(1+3\frac{\rho^2}{d^2}\right)=0 \qquad (80)$$

Equation 80 is corresponding to an equation obtained by eliminating ΔT between Eqs. 41 and 38. Substitution of ρ determined from Eq. 80 into Eq. 77 gives the value of ΔT at which the yielding starts outside.

The displacement u in $c \leq r \leq \rho$, as given by Eqs. 66 and 67 with Eqs. 7, 73, 75 and 76 and, assuming $T_d = 0°F$ and hence $k_d = k_0$, is

$$\frac{E}{2k_0} \frac{u}{c} = -\frac{r}{c} - \ln\frac{\rho}{r} + \nu\left(\frac{r}{c} - 1\right)$$

$$+ \frac{\beta\Delta T}{k_0} \frac{1}{\ln\frac{d}{c}}\left[\ln\frac{d}{c}\left(\frac{r}{c} + \ln\frac{\rho}{r}\right) - 2\left(\frac{\rho}{c} - \frac{r}{c}\right) + \ln\frac{\rho}{r} - \frac{r}{c}\ln\frac{r}{c}\right]$$

$$+ \nu\frac{\beta\Delta T}{k_0} \frac{1}{\ln\frac{d}{c}}\left[\left(1 + \ln\frac{d}{c}\right)\left(1 - \frac{r}{c}\right) + \frac{r}{c}\ln\frac{r}{c}\right]$$

$$+ \frac{E\alpha}{2\beta}\frac{\beta\Delta T}{k_0} \frac{1}{\ln\frac{d}{c}}\left(\frac{r}{c}\ln\frac{d}{c} - \frac{\rho}{c} + \frac{r}{c} - \frac{r}{c}\ln\frac{r}{c}\right) \tag{81}$$

Plastic Region Outside. Plastic region will form outside if Eq. 80 has a solution and which when substituted in Eq. 77 does not require ΔT to be such as to violate the assumptions regarding k. Stresses in the outer plastic region $\eta \leq r \leq d$ are given by Eqs. 43 and 44 with Eq. 73.

$$\frac{\sigma_r}{2k_d} = -\ln\frac{d}{r} + \frac{\beta\Delta T}{k_d} \frac{1}{2\ln\frac{d}{c}}\left(\ln\frac{d}{r}\right)^2 \tag{82}$$

$$\frac{\sigma_\theta}{2k_d} = 1 - \ln\frac{d}{r} + \frac{\beta\Delta T}{k_d} \frac{1}{2\ln\frac{d}{c}}\left[\left(\ln\frac{d}{r}\right)^2 - 2\ln\frac{d}{r}\right] \tag{83}$$

Stresses in the elastic region $\rho \leq r \leq \eta$, corresponding to Eqs. 47 and 48 may be obtained easily by using Eqs. 75, 76, 82 and 83.

The functions $f(\rho,\eta,\Delta T)$ and $g(\rho,\eta,\Delta T)$, corresponding to Eqs. 51 and 52 which determine ρ and η for a ΔT, are:

$$f(\rho,\eta,\Delta T) \equiv \frac{\rho}{c} + \frac{\beta\Delta T}{k_d}\frac{1}{\ln\frac{d}{c}}\left(-\frac{\rho}{c}\ln\frac{d}{c} - \frac{\rho}{c} + \frac{\rho^2}{c^2} + \frac{E\alpha}{4\beta}\frac{\rho^2}{c^2}\right)$$

$$+ \frac{\eta^2}{c^2} - \frac{\beta\Delta T}{k_d}\frac{\frac{\eta^2}{c^2}}{\ln\frac{d}{c}}\left(\ln\frac{d}{\eta} + \frac{E\alpha}{4\beta}\right) = 0 \tag{84}$$

$$g(\rho,\eta,\Delta T) \equiv 2-\frac{c}{\rho} + \frac{\beta\Delta T}{k_d}\frac{1}{\ln\frac{d}{c}}\left(\frac{c}{\rho}\ln\frac{d}{c} + \frac{c}{\rho} - 2\ln\frac{d}{\rho} - 1 + \frac{E\alpha}{4\beta}\ln\frac{\rho^2}{c^2}\right)$$

$$- 2\ln\frac{d}{\eta} + 1 + \frac{\beta\Delta T}{k_d}\frac{1}{\ln\frac{d}{c}}\left(-\ln\frac{d}{\eta} + \left(\ln\frac{d}{\eta}\right)^2 - \frac{E\alpha}{4\beta}\ln\frac{\eta^2}{c^2}\right) = 0 \tag{85}$$

The displacement u in $\eta \leq r \leq d$, as given by Eqs. 70 and 71 with Eqs. 7, 73, 82 and 83 and assuming $T_d = 0^\circ F$ and hence $k_d = k_0$, is

$$\frac{E}{2k_0}\frac{u}{r} = \ln\frac{r}{c} - \ln\frac{d}{c} + \frac{\eta^2}{r^2} + \nu\ln\frac{d}{r}$$

$$+ \frac{\beta\Delta T}{k_0}\frac{1}{2\ln\frac{d}{c}}\left[\left(\ln\frac{d}{r}\right)^2 - 2\frac{\eta^2}{r^2}\ln\frac{d}{\eta}\right]$$

$$- \nu\frac{\beta\Delta T}{k_0}\frac{1}{2\ln\frac{d}{c}}\left(\ln\frac{d}{r}\right)^2$$

$$+ \frac{E\alpha}{2\beta}\frac{\beta\Delta T}{k_0}\frac{1}{2\ln\frac{d}{c}}\left(2\ln\frac{d}{r} + 1 - \frac{\eta^2}{r^2}\right) \tag{86}$$

Program for a digital computer IBM 360/50 (using the PCP version of OS/360) was made by using the equations of this section to check the analysis numerically. For $\frac{d}{c} = 5$, by taking the values of $\frac{E\alpha}{4\beta}$ to be 3 and 100, the positions of the interfaces, stresses, strains and displacements were obtained for many values of $\frac{\beta\Delta T}{k_d}$ between 0 and 1. Also for $\frac{d}{c} = 10$ and $\frac{E\alpha}{4\beta} = 3$ the computer results were obtained; but as mentioned at the end of the section on inner plastic region, second yielding did not occur for $\frac{\beta\Delta T}{k_d} \leq 1$. Values of displacements and strains were calculated assuming $T_d = 0^\circ F$.

10. General Discussion of Case $\Delta T < 0$.

As shown before (Figure 4), under general given conditions, the uncertainty of place(s) of the development of plastic region(s) arise right from the initial yielding of the ring; and hence the analysis in general cannot be made in a definite order or led to completion. Only a specific problem can be solved completely and in definite order following the general procedure for the case of $\Delta T > 0$ unless the plastic regions unload as the loading progresses requiring a modified analysis. Analysis must be made in small steps for decreasing ΔT carefully checking the occurrence of a new independent yielding and retreat of the interfaces.

The yield conditions of the inner and outer plastic regions could only be as represented by lines AB and EF (Figure 9) respectively for $\Delta T > 0$; and in the same way they would only be as represented by lines BC and DE respectively for $\Delta T < 0$. These yield conditions together with the known boundary conditions: $\sigma_{rc} = \sigma_{rd} = 0$ completely determine the stresses in those plastic regions.

For $\Delta T < 0$, there are possibilities that one or more plastic regions, with an elastic region separating each other, may appear besides the inner and outer ones, for example as shown in Figure 4. Such a plastic region may appear with one of the three yield conditions represented by lines BC, CD and DE (Figure 9). Lines AB, AF and EF requiring σ_r to be negative seem highly unlikely assuming, as the plastic regions develop with increasing loading $(-\Delta T)$, σ_r may not decrease after it has been positive throughout the initially entirely elastic ring and has been increasing with $-\Delta T$ up to the first yielding (Eq. 15 and Figure 2). The stresses are statically determinate in a plastic region appearing with a yield condition $\sigma_r = 2k$ (line CD), but they are not in the one appearing with either $\sigma_\theta = 2k$ (line BC) or $\sigma_r - \sigma_\theta = 2k$ (line DE) because of the absence of a known boundary condition to determine the constant of integration of Eq. 1 used with the yield condition. In such case, displacement considerations must be made on account of the two known boundary conditions for displacement u: $\frac{u}{r} = \epsilon_\theta$ must be elastic at the two interfaces, and one unknown constant of integration involved in obtaining u just like that in Eq. 67 or 70. Thus the stresses in such plastic regions would be dependent of the positions of the interfaces.

Having determined the stresses -- dependent or independent of the positions of relevant interfaces -- in the plastic regions, the procedure to determine the positions of the interfaces, stresses in the elastic regions and the displacements throughout the ring is the same as that applied for $\Delta T > 0$. The plastic stresses at the interfaces may

not satisfy any of the four conditions given at the end of the section
on the behavior of stresses in an elastic region, but the behavior of
the elastic stresses can be inferred by some reasoning and the question
of violation of yield conditions in the elastic regions can be settled.

11. Conclusions.

The problem is comprised of two cases: (a) $\Delta T > 0$ and (b) $\Delta T < 0$,
depending upon whether the inside temperature is higher or lower than
the outside temperature. Conclusions for both the cases are different
and, hence, they are presented separately.

(a) $\underline{\Delta T > 0}$: The initial yielding always occurs inside, and a
small finite plastic region with an elastic-plastic interface ρ
develops inside due to a small increment in ΔT. As ΔT keeps increas-
ing, ρ moves definitely outward if T_d is held constant, whereas it may
not if T_c is held constant. If ρ keeps increasing, before it reaches
the outer boundary d, a second (outer) plastic region with an elastic-
plastic interface η is likely to develop from d. No third, new plastic
region can develop as the inner and outer plastic regions approach
each other; and ΔT is required to approach infinity to let $\frac{\eta}{\rho} \to 1$ for
complete yielding. If the outer plastic region does not develop at
all and ρ keeps increasing, then also the ring tends to become complete-
ly plastic if ΔT approaches infinity. However, the assumptions made
for function k (Inequalities 17 or 18) will mostly impose limit on ΔT
in a problem which is being solved from actual physical data and com-
plete yielding will not be realized. The displacement solution provides

a justification to the a priori assumption of infinitesimal strains at high temperatures or at high difference in the boundary temperatures.

It is shown that in spite of increasing loading (ΔT), the plastic regions (inner only when alone, or both or either of inner and outer) may suffer unloading, i.e. the elastic-plastic interfaces may retreat. This is perhaps a characteristic unique of the temperature problems in which the yield stress decreases (an effect of softening) or increases (an effect of hardening) as temperature increases or decreases respectively, or in which the softening or hardening effects are produced by the thermal dependence of physical parameters. If a part of an elastic region had been in plastic state before, the analysis for that part must include the residual strains due to the previous plastic deformation; and thus, as a plastic region starts to unload, the stress and displacement analyses are required to be carried out together.

(b) $\underline{\Delta T < 0}$: The elastic-plastic behavior of rings in this case is in general different from that discussed above. Step by step analysis of the problem under general given conditions cannot be made; each specific problem must be treated in its own right and conclusions be drawn following the general procedure adopted for the case of $\Delta T > 0$. More than two plastic regions are likely to develop. Stresses would not be statically determinate in an independent plastic region, except inner or outer one, occurring with the yield condition either $\sigma_\theta = 2k$ or $\sigma_r - \sigma_\theta = 2k$. It is possible that the interfaces may retreat despite increasing loading.

The present problem when extended by allowing three more
considerations may yield some more interesting results. Unloading
(if occurring) of plastic regions with increasing loading, unloading
of the ring leading to residual stresses and the transient tempera-
ture distributions while loading and unloading the ring may be
considered.

References

1. Cowper, G. R., "The Elastoplastic Thick-Walled Sphere Subjected to a Radial Temperature Gradient," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, series, E, pp. 496-500, 1960.

2. Johnson, W. and P. B. Mellor, "Elastic-Plastic Behavior of a Thick-Walled Sphere of Non-Workhardening Material Subject to Steady State Radial Temperature Gradient," International Journal of Mechanical Sciences, vol. 4, pp. 147-158, 1962.

3. Rogozinski, M., "Some Problems of Thermoplasticity of a Spherical Shell," Non-homogeneity in Elasticity and Plasticity, Ed. W. Olszak, Pergamon Press, New York, pp. 215-226, 1959.

4. Bland, D. R., "Elastoplastic Thick-Walled Tubes of Work-Hardening Material Subjected to Internal and External Pressures and to Temperature Gradients," Journal of the Mechanics and Physics of Solids, vol. 4, pp. 209-229, 1956.

5. Kammash, T. B., S. A. Murch and P. M. Naghdi, "The Elastic-Plastic Cylinder Subjected to Radially Distributed Heat Source, Lateral Pressure and Axial Force with Application to Nuclear Reactor Fuel Elements," Journal of the Mechanics and Physics of Solids, vol. 8, pp. 1-25, 1960.

6. Wilhoit, Jr., J. C., "Elastic-Plastic Stresses in Rings Under Steady State Radial Temperature Variation," Proc. of the Third U.S. National Congress of Applied Mechanics, pp. 693-700, 1958.

7. Yuksel, H., "Elastic Plastic Stresses in Free Plate with Periodically Varying Temperature," Journal of Applied Mechanics, vol. 25, Trans. ASME, vol. 80, series E, pp. 603-606, 1958.

8. Landau, H. G., J. H. Weiner, and E. E. Zwicky, Jr., "Thermal Stress in a Viscoelastic-Plastic Plate with Temperature Dependent Yield Stress," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, series E, pp. 297-302, 1960.

9. Mendelson, A. and S. W. Spero, "A General Solution for the Elastoplastic Thermal Stresses in a Strain-Hardening Plate with Arbitrary Material Properties," Journal of Applied Mechanics, vol. 29, Trans. ASME, vol. 84, series E, pp. 151-158, 1962.

10. Parkus, H., "Stresses in a Centrally Heated Disc," Proc. of the Second U.S. National Congress of Applied Mechanics, pp. 307-311, 1954.

11. Landau, H. G. and E. E. Zwicky, Jr., "Transient and Residual Thermal Stresses in an Elastic-Plastic Cylinder," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, series E, pp. 481-488, 1960.

12. Lee, Y. C. and W. Jaunzemis, "Elastic-Plastic Response of a Slab to a Heat Pulse," Tech. Rep. No. 6, Contract AF-AFOSR 128-63, The Pennsylvania State University, University Park, Pa., 1963.

13. Prager, W., "Non-Isothermal Plastic Deformation," Proc. Koninkl. Nederl. Akad. van Wetenschappen, series B, vol. 61, no. 3, pp. 178-182, 1958.

14. Boley, B. A. and J. H. Weiner, "Theory of Thermal Stresses," John Wiley and Sons, Inc., New York, pp. 453-489, 1960.

15. Koiter, W. T., "On Partially Plastic Thick-Walled Tubes," Biezeno Anniversary Volume, Haarlem, Holland, pp. 232-251, 1953.
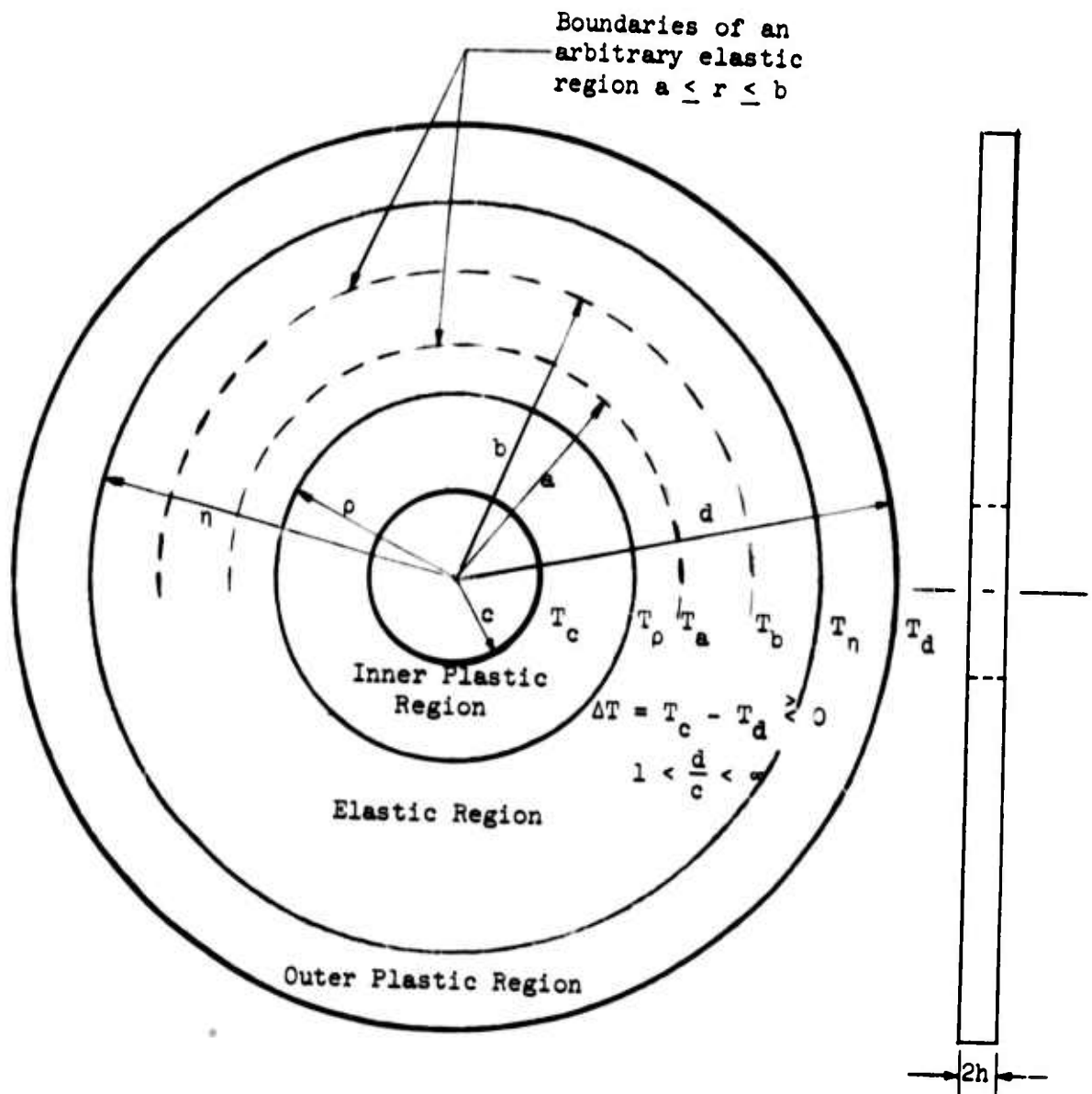
Figure 1 - Various Zones in a Ring $c \leq r \leq d$.
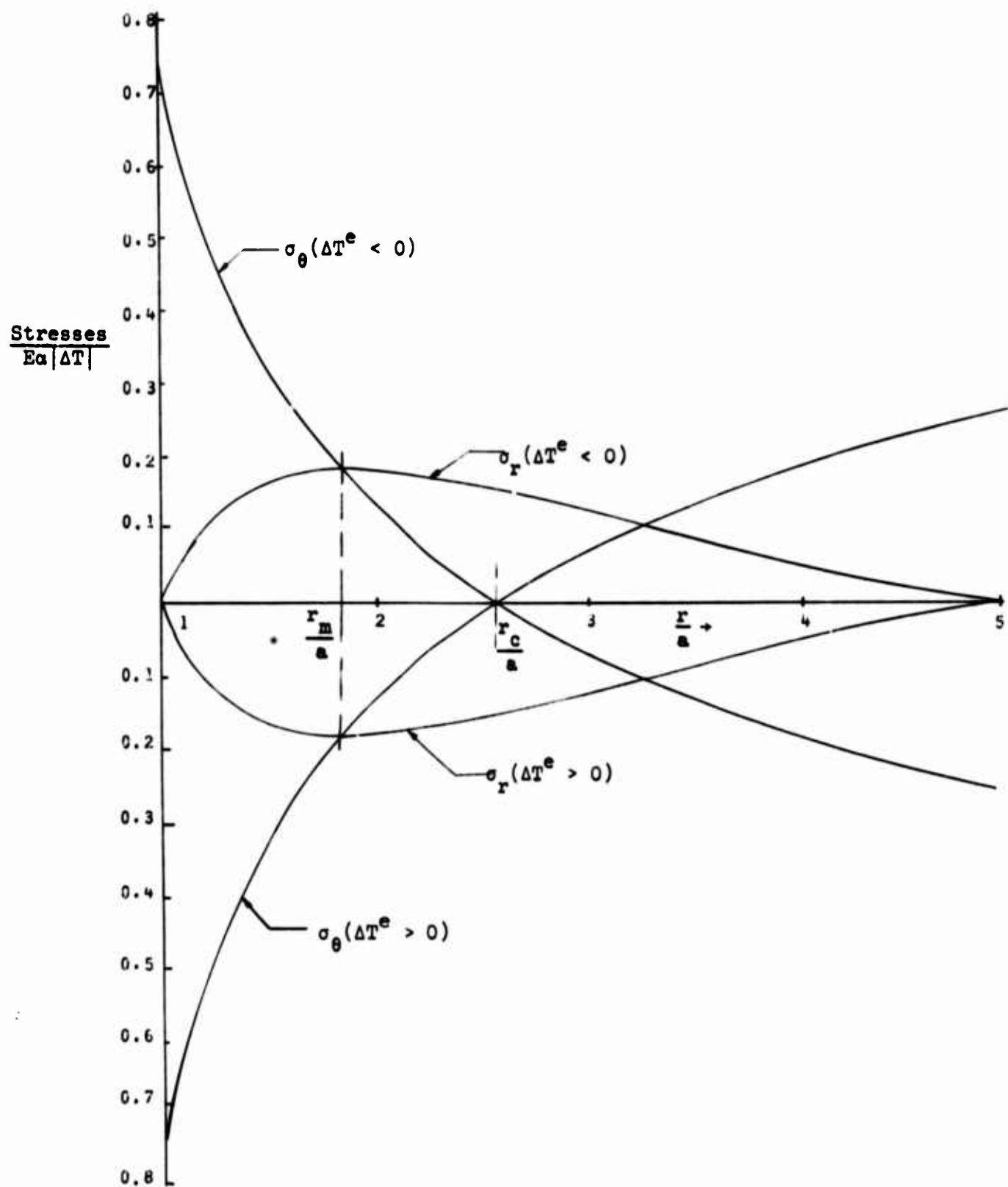
Figure 2 - Typical Elastic Stress Distribution When
$$p_a = p_b = 0 \ \left(\frac{b}{a} = 5\right)$$
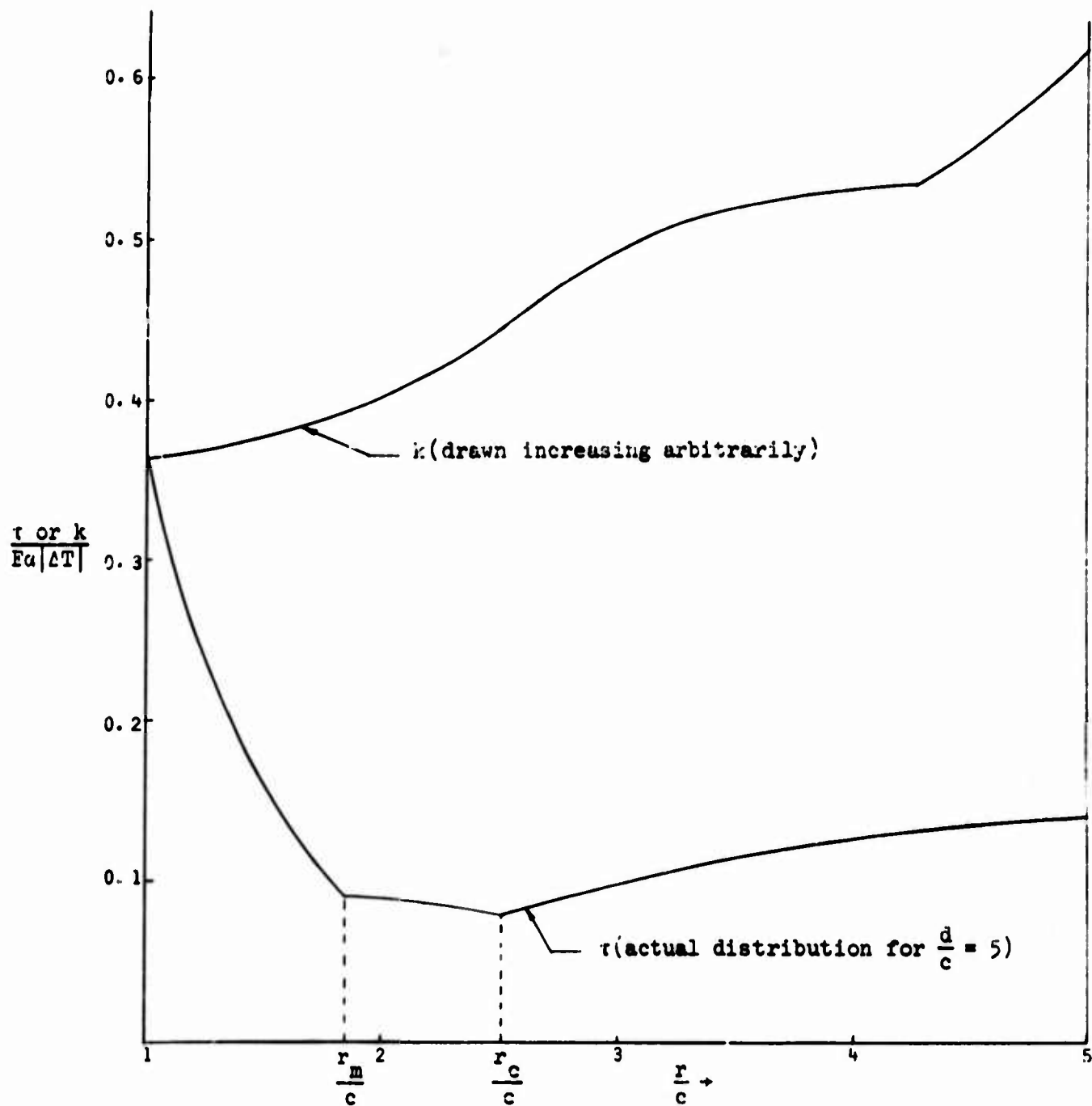
Figure 3 - Incipience of Yielding When $\Delta T > 0$.
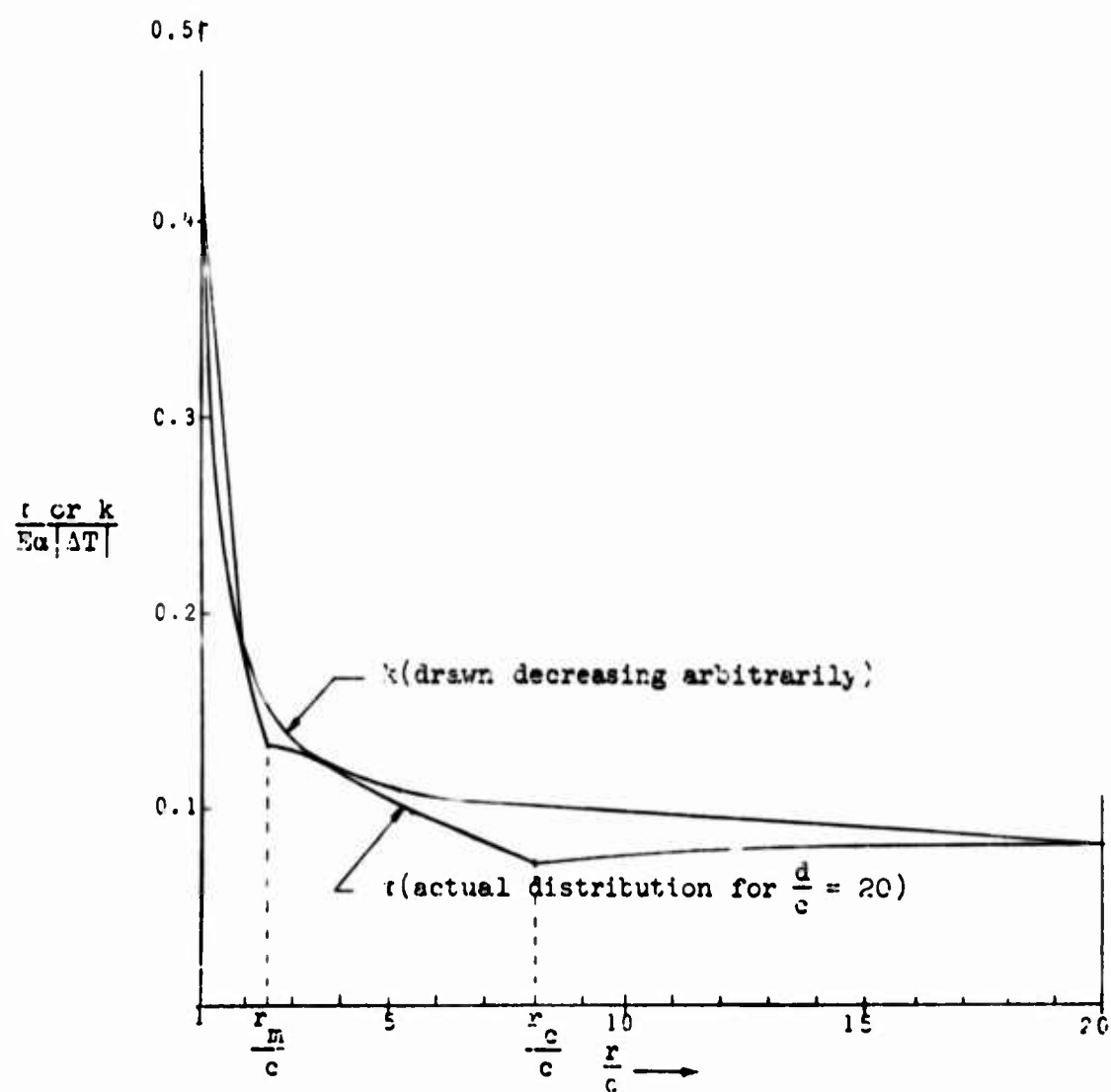
Figure 4 - Incipient Multiple Yieldings When ΔT < 0.
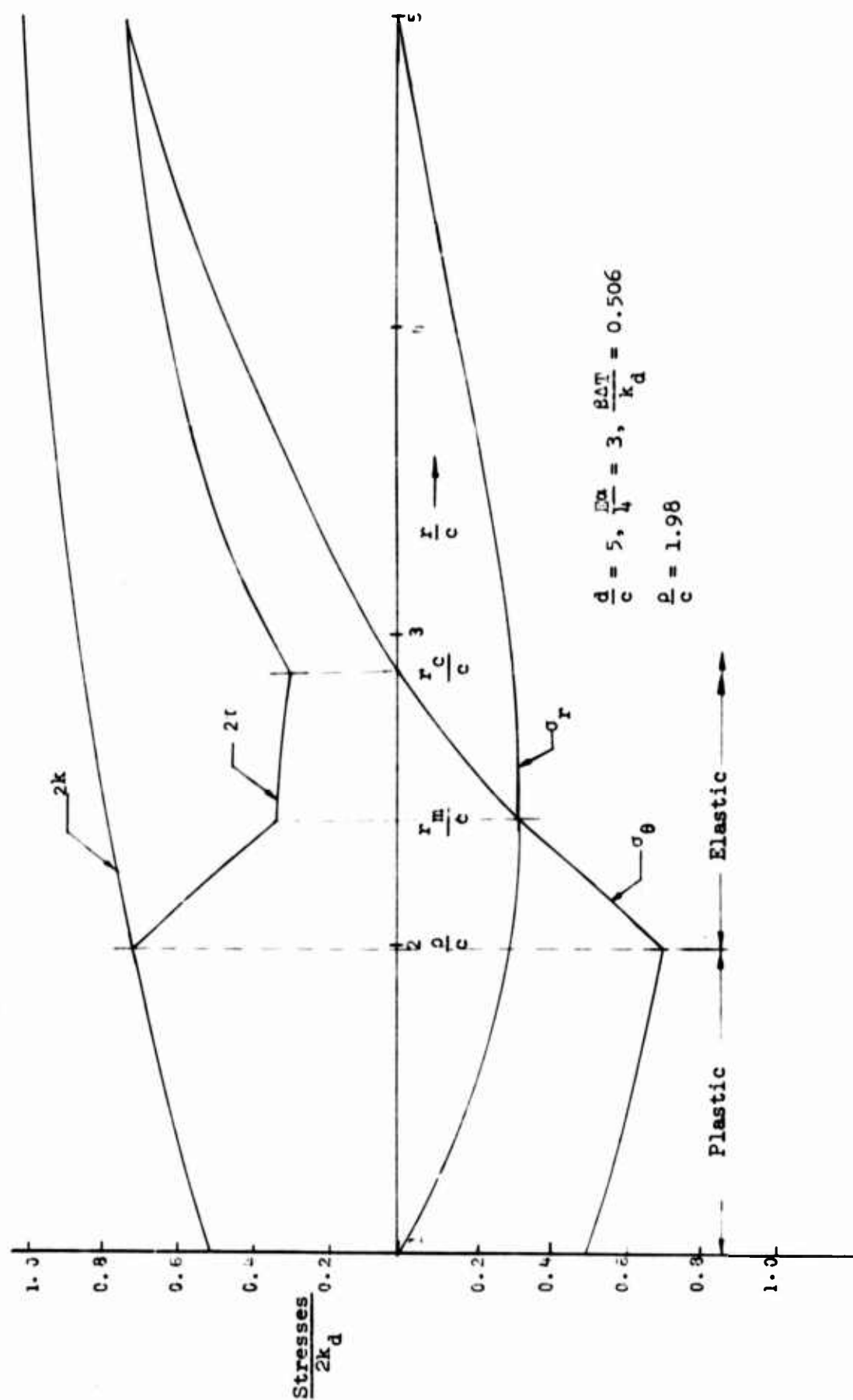
Figure 5 – Typical Distribution of Stresses When Inner Plastic Region Exists.

$$\frac{d}{c} = 5, \quad \frac{Ea}{4\beta} = 3, \quad \frac{\beta\Delta T}{k_d} = 0.973$$

$$\frac{\rho}{c} = 2.67, \quad \frac{\eta}{c} = 4.59$$
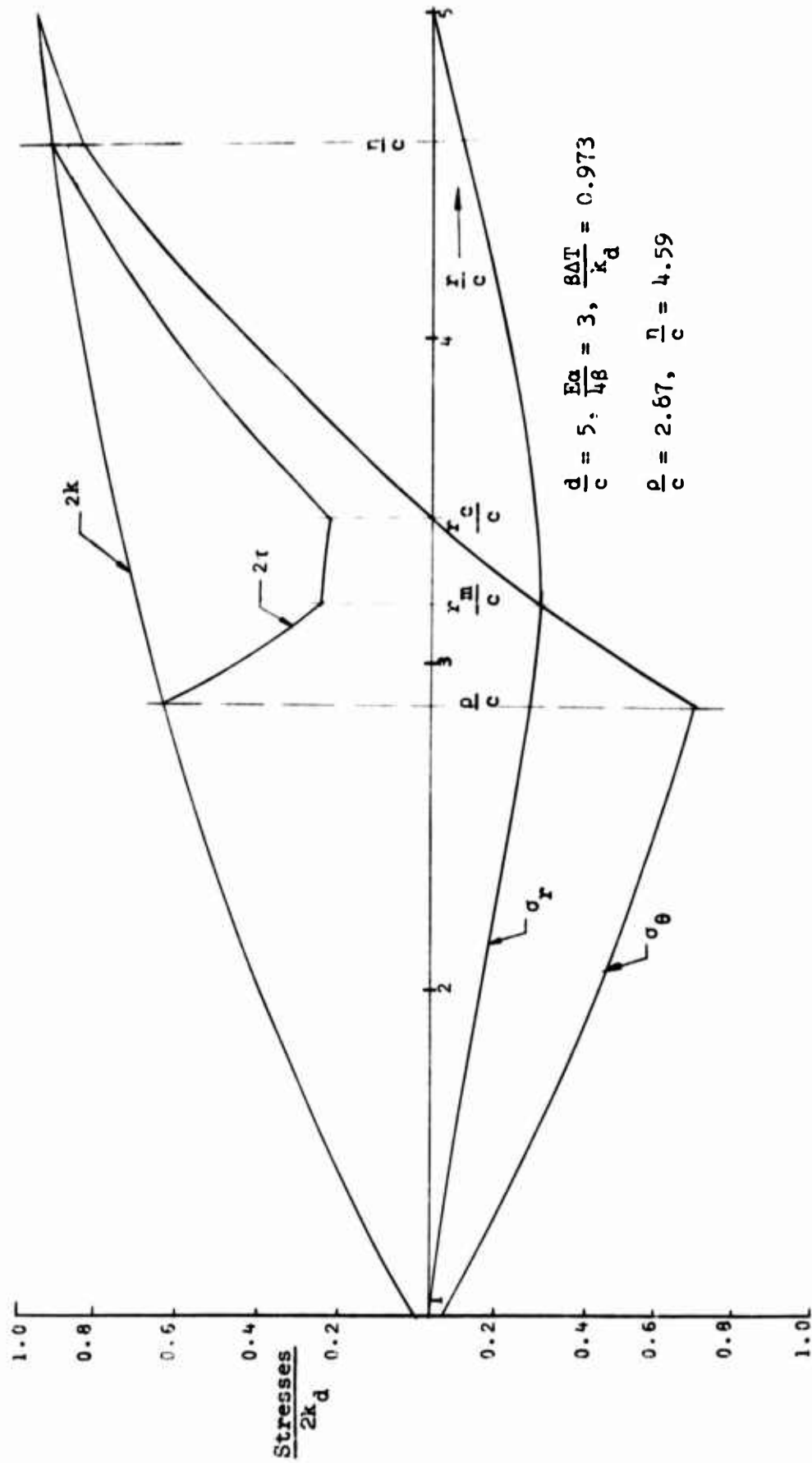
Figure 6 – Typical Distribution of Stresses When Inner and Outer Plastic Regions Exist.
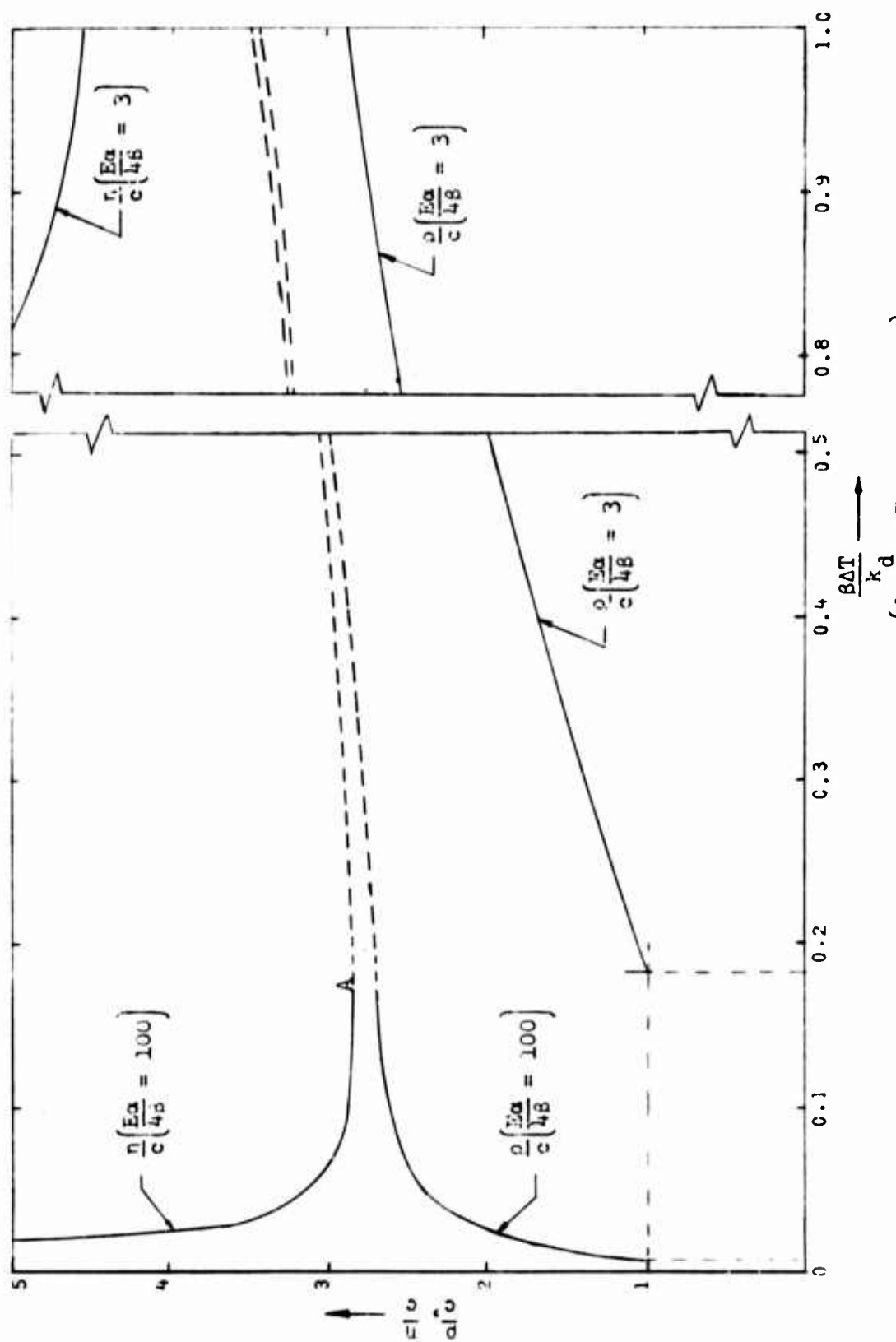
Figure 7 – ρ and η vs. ΔT $\left(\frac{d}{c} = 5, \frac{E\alpha}{4\beta} = 3 \text{ and } 100 \right)$
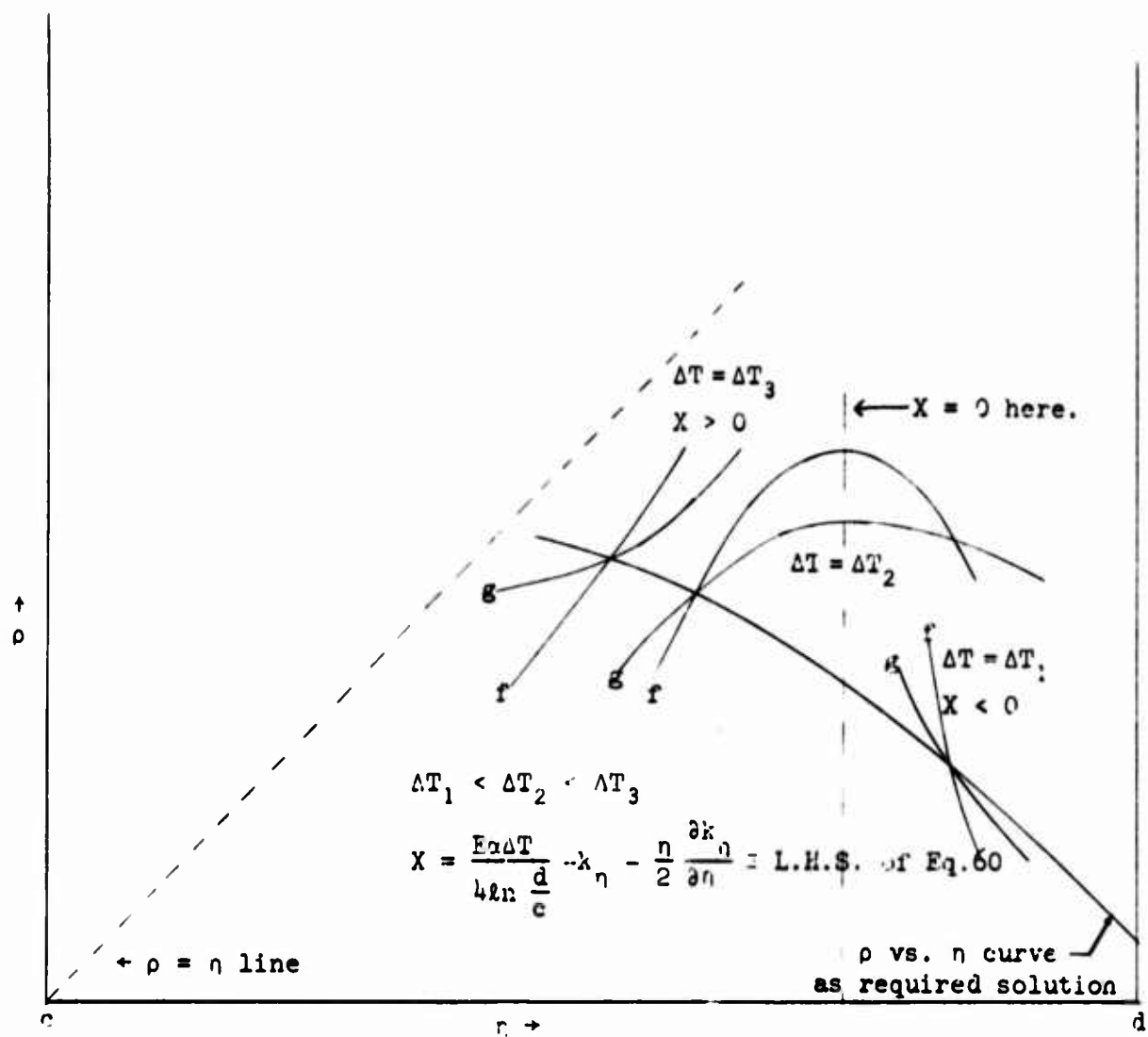
Figure 8 - Curves f and g in ηρ Plane.
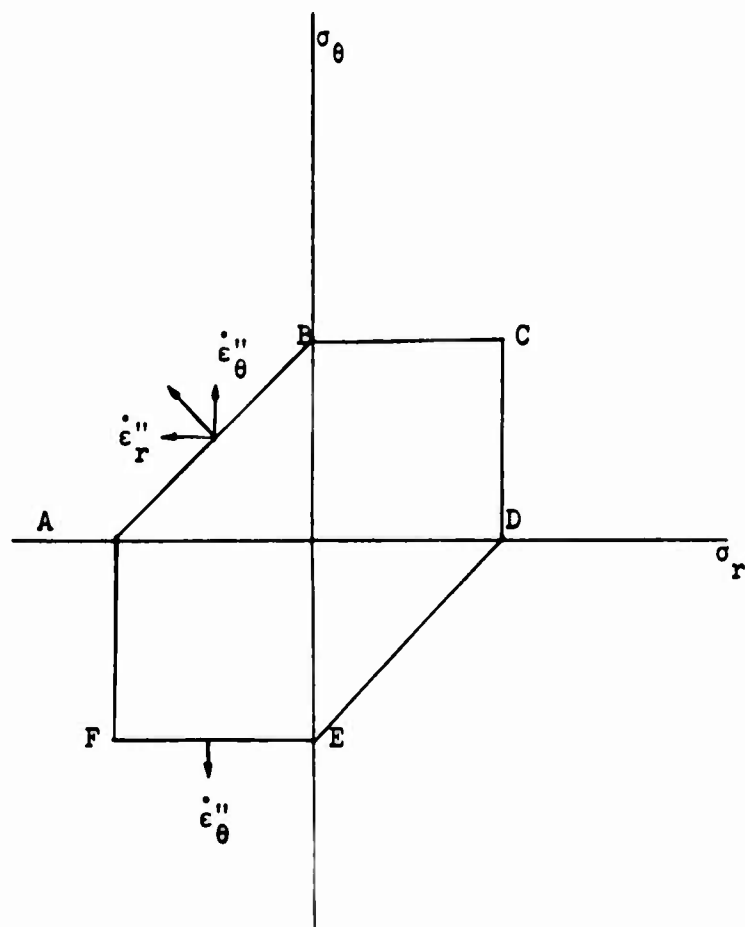
Figure 9 - Tresca Yield Condition and
Strain Rate Vectors.

Ratio of strains and displacement to appropriate constants

8.0

6.0

$\epsilon_r \times \dfrac{E}{2k_d}$

$\leftarrow \left(\epsilon_r' \text{ or } \epsilon_\theta'\right) \times \dfrac{E}{2k_d}$ due to temperature only, assuming $k_d = k_o = $ constant

$\underline{\epsilon_r'' = 0}$

4.0

$\leftarrow \epsilon_r' \times \dfrac{10E}{k_d}$ due to stresses only

2.0

Incipience of yielding

yielding starts outside $\rightarrow$

0

0.1 0.2   0.4   0.6   0.8   1.0

$\dfrac{\beta\Delta T}{k_d} \rightarrow$

2.0

$\leftarrow \epsilon_\theta \times \dfrac{E}{2k_d} = u \times \dfrac{E}{2k_d c}$

$\leftarrow \epsilon_\theta' \times \dfrac{5E}{k_d}$ due to stresses only

4.0

6.0

8.0

$\leftarrow \epsilon_\theta'' \times \dfrac{E}{2k_d}$

1.0
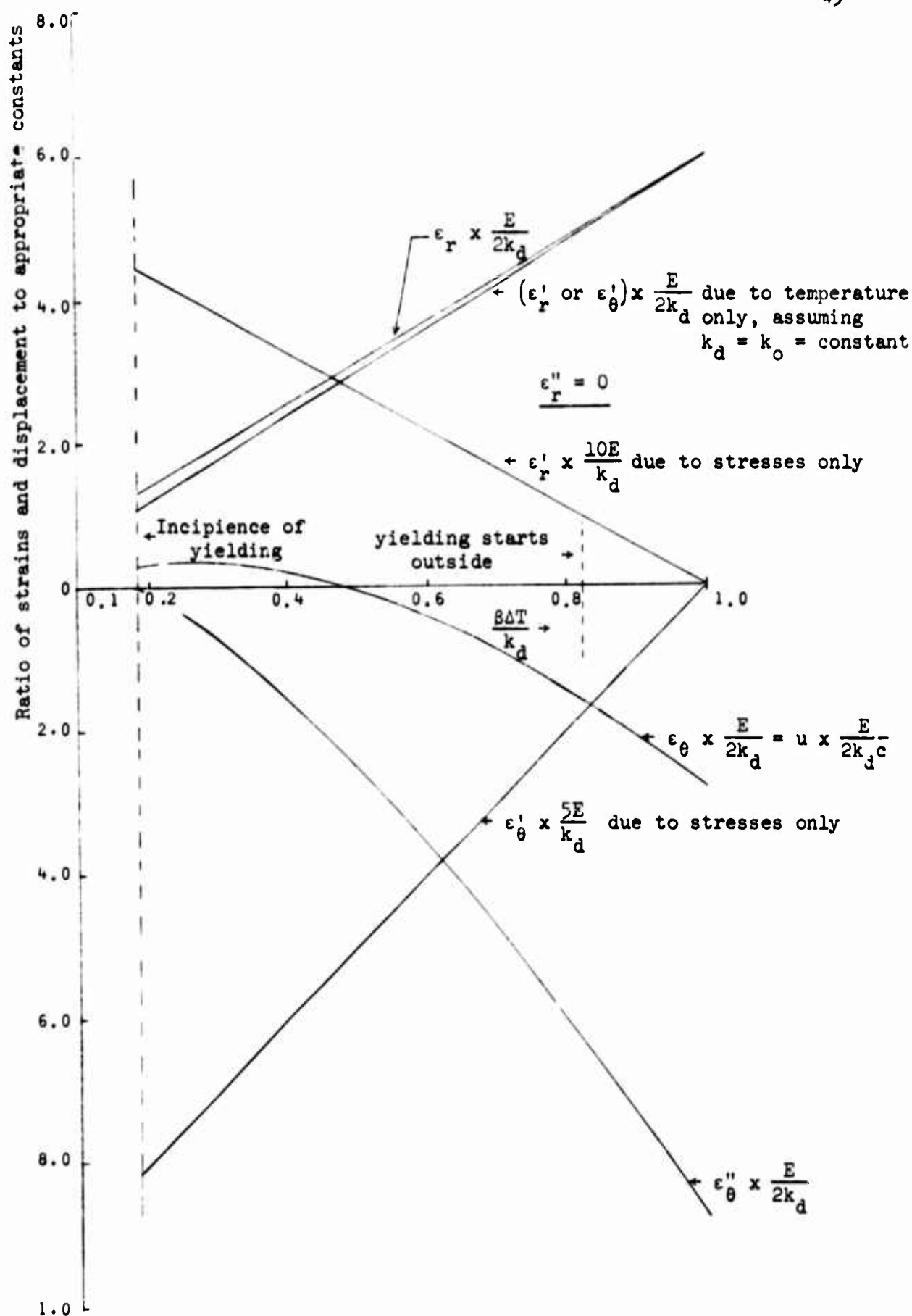
Figure 10 - Strains and Displacement vs. ΔT at $r = c$ $\left(\dfrac{d}{c} = 5, \dfrac{E\alpha}{4\beta} = 3\right)$
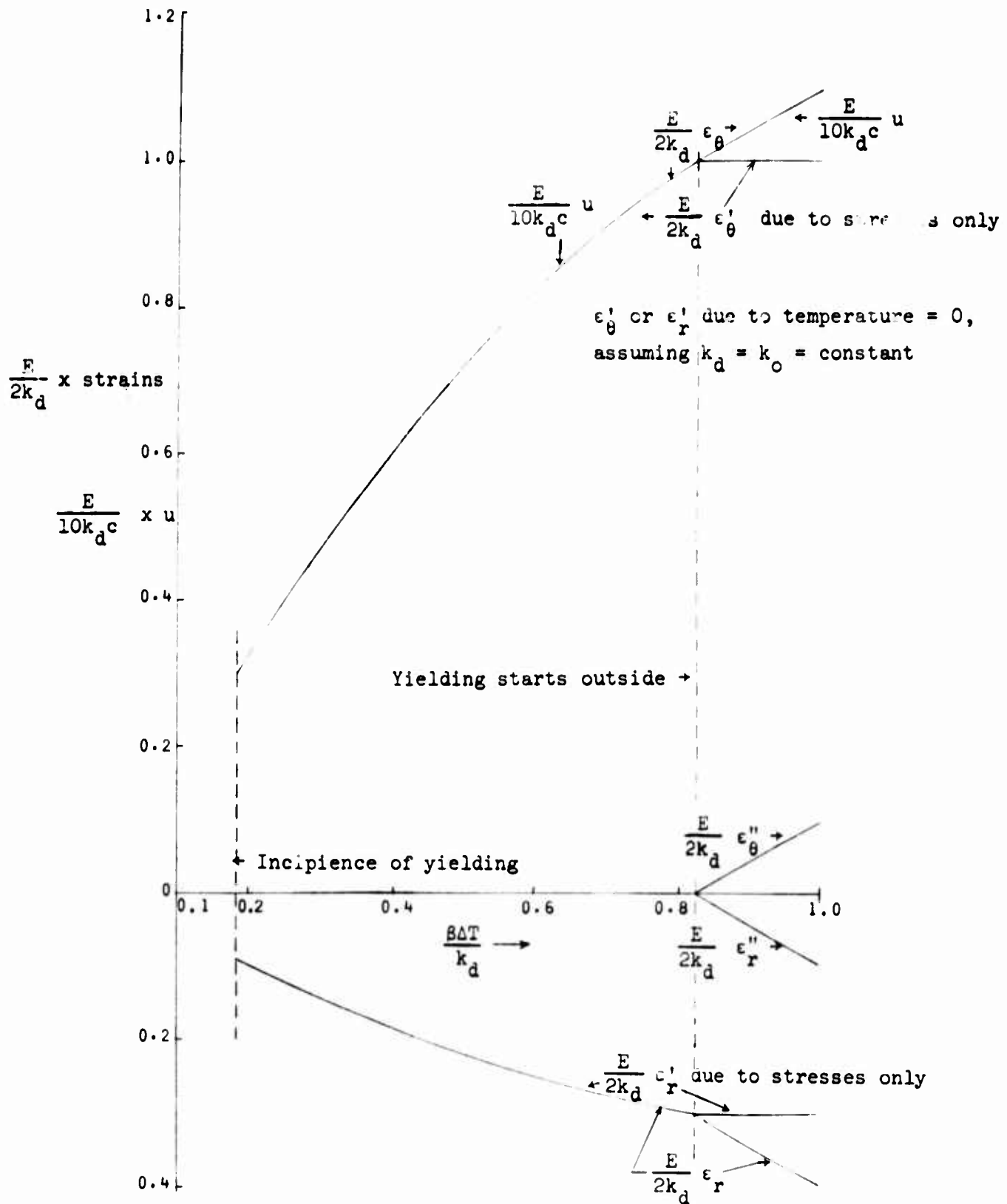
Figure 11 - Strains and Displacement vs. ΔT at $r = d$ $\left(\dfrac{d}{c} = 5, \dfrac{E\alpha}{4\beta} = 3\right)$

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
The Pennsylvania State University Department of Engineering Mechanics University Park, Pennsylvania 16802	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

AN ELASTO-PLASTIC ANALYSIS OF CIRCULAR RINGS WITH TEMPERATURE DEPENDENT YIELD STRESS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific   Interim

5. AUTHOR(S) *(First name, middle initial, last name)*

N. T. Patel
L. W. Hu

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1967	50	15

8a. CONTRACT OR GRANT NO. AF-AFOSR-127-67	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 9782-02	
c. 61445014	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d. 681307	AFOSR 67-2836

10. DISTRIBUTION STATEMENT

1. Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
TECH, OTHER	AF Office of Scientific Research (SREM) 1400 Wilson Boulevard Arlington, Virginia 22209

13. ABSTRACT

The elastic-plastic behavior of a ring under steady state radial temperature gradient is analyzed. The material is assumed to be elastic-perfectly plastic and its yield stress in simple shear to be a continuous and general monotonically decreasing positive function of temperature. Modified Tresca's yield condition and the associated flow rules are used.

DD FORM NOV 65 **1473**

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Thermal Stress						
Elasto-Plastic Analysis						
Circular Rings						
Temperature Dependent Yield Stress						